Factors of Natural Numbers

To find the factors of a natural number, express it as a product of two natural numbers. For example, consider 30:

$$30 = 1 \times 30 = 2 \times 15 = 3 \times 10 = 5 \times 6$$

Thus, the factors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30. Among these, 2, 3, and 5 are prime factors because they are prime numbers that divide 30 exactly.

Note that 1 is a factor of every natural number, but when expressing numbers as products of factors, 1 is usually omitted unless specifically required.

A number written as a product of its prime factors is said to be in prime factor form. For example:

- 30 = 2 \times 3 \times 5

- 70 = 2 \times 5 \times 7
- 90 = 2 \times 3 \times 3 \times 5

This concept extends to algebraic expressions, where expressions can be expressed as products of their factors.

Worked Example

Express 30 in prime factor form:

Step 1: Find prime factors of 30: 2, 3, 5

Step 2: Write 30 as product of prime factors:

$$30 = 2 \times 3 \times 5$$

Practice Set

- List all factors of 24 and write its prime factor form.
- Find prime factors of 56 and express it as a product of prime factors.
- Explain why 1 is not usually written as a factor in prime factorization.

Answer Key

- Factors of 24: 1, 2, 3, 4, 6, 8, 12, 24; Prime factor form: $2 \times 2 \times 2 \times 3$
- Prime factors of 56: 2, 7; Prime factor form: $2 \times 2 \times 2 \times 7$
- 1 is omitted because multiplying by 1 does not change the value, so it is redundant in factorization.

Quick Reference

- Factors divide the number exactly without remainder.
- Prime factors are prime numbers dividing the number.
- Prime factorization expresses a number as product of prime factors.

Glossary

- **Factor:** A number that divides another number exactly.
- **Prime Factor:** A prime number that divides a number exactly.
- **Prime Factorization:** Expressing a number as a product of prime factors.

Factors of Algebraic Expressions

In algebraic expressions, terms are products of factors. For example, the term $5xy$ is formed by factors 5, x, and y:

$$5xy = 5 \times x \times y$$

Note that 1 is a factor of every term, but it is usually not written explicitly.

Factors that cannot be further expressed as products of factors are called irreducible factors (analogous to prime factors in numbers). For example, 5, x, and y are irreducible factors of $5xy$.

Consider the expression $3x(x+2)$. Its irreducible factors are 3, x, and $x+2$:

$$3x(x+2) = 3 \times x \times (x+2)$$

Similarly, $10x(x+2)(y+3)$ can be expressed as:

$$10x(x+2)(y+3) = 2 \times 5 \times x \times (x+2) \times (y+3)$$

Worked Example

Express $5xy$ in irreducible factor form:

$$5xy = 5 \times x \times y$$

Practice Set

- Express $12a^2b$ in irreducible factor form.
- Write $7m(n+1)$ as product of irreducible factors.
- Explain why xy is not irreducible.

Answer Key

- $12a^2b = 2 \times 2 \times 3 \times a \times a \times b$
- $7m(n+1) = 7 \times m \times (n+1)$
- Because $xy = x \times y$, it can be further factored, so it is not irreducible.

Quick Reference

- Terms are products of factors.
- Irreducible factors cannot be factored further.

- 1 is a factor but usually omitted.

Glossary

- **Term:** A product of factors in an algebraic expression.
- **Irreducible Factor:** A factor that cannot be factored further.

Factorisation

Factorisation is the process of expressing an algebraic expression as a product of its factors, which may be numbers, variables, or algebraic expressions.

Expressions like $3xy$, $5x^2y$, $2x(y+2)$, and $5(y+1)(x+2)$ are already in factorised form.

Expressions like $2x+4$, $3x+3y$, x^2+5x, and x^2+5x+6 require systematic methods to find their factors.

Method of Common Factors

To factorise using common factors:

1. Write each term as a product of irreducible factors.
2. Identify common factors in all terms.
3. Use distributive law to factor out the common factors.

Example: Factorise $2x+4$

$$2x = 2 \times x, \quad 4 = 2 \times 2$$

Common factor is 2:

$$2x + 4 = 2 \times x + 2 \times 2 = 2(x + 2)$$

Worked Example 1

Factorise $5xy + 10x$

$$5xy = 5 \times x \times y, \quad 10x = 2 \times 5 \times x$$

Common factors: 5 and x

$$5xy + 10x = 5x(y + 2)$$

Practice Set

- Factorise $12a^2b + 15ab^2$
- Factorise $10x^2 - 18x^3 + 14x^4$
- Factorise $22y - 33z$

Answer Key

- $12a^2b + 15ab^2 = 3ab(4a + 5b)$
- $10x^2 - 18x^3 + 14x^4 = 2x^2(5 - 9x + 7x^2)$
- $22y - 33z = 11(2y - 3z)$

Quick Reference

- Identify common factors.
- Use distributive law to factor out common factors.

Glossary

- **Common Factor:** A factor common to all terms.
- **Distributive Law:** $a(b+c) = ab + ac$

Factorisation by Regrouping Terms

When no single common factor exists for all terms, group terms to find common factors within groups.

Example: Factorise $2xy + 2y + 3x + 3$

Group terms:

$$(2xy + 2y) + (3x + 3)$$

Factor each group:

$$2y(x+1) + 3(x+1)$$

Common factor $x+1$:

$$(x+1)(2y+3)$$

Regrouping means rearranging terms to form groups with common factors.

Worked Example

Factorise $6xy - 4y + 6 - 9x$

Group terms:

$$(6xy - 4y) + (6 - 9x)$$

Factor each group:

$$2y(3x-2) - 3(3x-2)$$

Common factor $3x - 2$:

$$(3x-2)(2y-3)$$

Practice Set

- Factorise $4ab + 8a + 3b + 6$
- Factorise $5xy + 10y + 3x + 6$

Answer Key

- $4ab + 8a + 3b + 6 = (4a + 3)(b + 2)$
- $5xy + 10y + 3x + 6 = (5y + 3)(x + 2)$

Quick Reference

- Group terms to find common factors.
- Factor each group.
- Factor out common binomial.

Glossary

- **Regrouping:** Rearranging terms to facilitate factorisation.

Factorisation Using Identities

Use algebraic identities to factorise expressions matching their forms.

Key identities:

Identity	Expression	Factorised Form
(I)	$(a+b)^2 = a^2 + 2ab + b^2$	$(a+b)^2$
(II)	$(a-b)^2 = a^2 - 2ab + b^2$	$(a-b)^2$
(III)	$(a+b)(a-b) = a^2 - b^2$	$(a+b)(a-b)$

(IV)	$(x+a)(x+b) = x^2 + (a+b)x + ab$	$(x+a)(x+b)$

Worked Example 1

Factorise $x^2 + 8x + 16$

Recognise as $a^2 + 2ab + b^2$ with $a = x, b = 4$:

$$x^2 + 8x + 16 = (x+4)^2$$

Worked Example 2

Factorise $49p^2 - 36$

Recognise as difference of squares:

$$49p^2 - 36 = (7p)^2 - 6^2 = (7p-6)(7p+6)$$

Practice Set

- Factorise $4y^2 - 12y + 9$
- Factorise $a^2 - 2ab + b^2 - c^2$
- Factorise $m^4 - 256$

Answer Key

- $4y^2 - 12y + 9 = (2y - 3)^2$
- $a^2 - 2ab + b^2 - c^2 = (a - b - c)(a - b + c)$
- $m^4 - 256 = (m - 4)(m + 4)(m^2 + 16)$

Quick Reference

- Match expression to identity form.
- Apply corresponding factorisation.

Glossary

- **Perfect Square:** An expression of the form a^2.
- **Difference of Squares:** Expression of the form $a^2 - b^2$.

Factors of the Form (x + a)(x + b)

Expressions like $x^2 + 5x + 6$ can be factorised as $(x + a)(x + b)$ where:

$$ab = \text{ constant term} \quad \text{and} \quad a + b = \text{ coefficient of } x$$

Example: Factorise $x^2 + 5x + 6$

Find a, b such that:

$$ab = 6, \quad a + b = 5$$

Try $a = 2, b = 3$ since $2 \times 3 = 6$ and $2 + 3 = 5$.

Therefore,

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

Worked Example

Factorise $y^2 - 7y + 12$

Find a, b such that:

$$ab = 12, \quad a + b = 7$$

Try $a = 3, b = 4$ since $3 \times 4 = 12$ and $3 + 4 = 7$.

Rewrite expression:

$$y^2 - 7y + 12 = y^2 - 3y - 4y + 12 = y(y - 3) - 4(y - 3) = (y - 3)(y - 4)$$

Practice Set

- Factorise $z^2 - 4z - 12$
- Factorise $3m^2 + 9m + 6$

Answer Key

- $z^2 - 4z - 12 = (z-6)(z+2)$
- $3m^2 + 9m + 6 = 3(m+1)(m+2)$

Quick Reference

- Find two numbers a, b such that $ab =$ constant term and $a + b =$ coefficient of x.
- Rewrite expression and factor by grouping.

Glossary

- **Coefficient:** Numerical factor of a term.
- **Constant Term:** Term without variable.

Division of Algebraic Expressions

Division is the inverse of multiplication. For algebraic expressions:

$$\text{Dividend} = \text{Divisor} \times \text{Quotient}$$

We consider division where the remainder is zero.

Division of a Monomial by Another Monomial

Example: Divide $6x^3$ by $2x$

Express in irreducible factors:

$$6x^3 = 2 \times 3 \times x \times x \times x, \quad 2x = 2 \times x$$

Cancel common factors:

$$\frac{6x^3}{2x} = \frac{2 \times 3 \times x \times x \times x}{2 \times x} = 3x^2$$

Division of a Polynomial by a Monomial

Example: Divide $4y^3 + 5y^2 + 6y$ by $2y$

Divide each term:

$$\frac{4y^3}{2y} + \frac{5y^2}{2y} + \frac{6y}{2y} = 2y^2 + \frac{5}{2}y + 3$$

Division of a Polynomial by a Polynomial

Factorise numerator and denominator and cancel common factors.

Example: Divide $7x^2 + 14x$ by $x + 2$

Factor numerator:

$$7x^2 + 14x = 7x(x + 2)$$

Divide:

$$\frac{7x(x + 2)}{x + 2} = 7x$$

Practice Set

- Divide $24xy^2z^3$ by $6yz^2$
- Divide $63a^2b^4c^6$ by $7a^2b^2c^3$
- Divide $44(x^4 - 5x^3 - 24x^2)$ by $11x(x - 8)$

Answer Key

- $4xyz$
- $9b^2c^3$
- $4x(x + 3)$

Quick Reference

- Express terms in irreducible factors.
- Cancel common factors.
- Divide each term when dividing by monomial.

Glossary

- **Dividend:** Expression being divided.
- **Divisor:** Expression dividing the dividend.
- **Quotient:** Result of division.