Proportionality and Ratios

Proportionality describes a relationship where two or more quantities change by the same factor. Ratios express this relationship as a comparison of quantities. For example, a ratio of 2 : 1 means for every 2 units of the first quantity, there is 1 unit of the second.

Formula Derivation

Two ratios $a : b$ and $c : d$ are proportional if their cross products are equal:

$$a \times d = b \times c$$

Equivalently,

$$\frac{a}{c} = \frac{b}{d}$$

Worked Illustration

Viswanath's mixture: 6 cups rice : 3 cups urad dal = 6 : 3

Puneet's mixture: 4 cups rice : 2 cups urad dal = 4 : 2

Check proportionality by cross multiplication:

$$6 \times 2 = 12, \quad 3 \times 4 = 12$$

Since both products are equal, the ratios are proportional.

Solved Example

Verify if the ratios 8 : 12 and 4 : 6 are proportional.

Cross multiply:

$$8 \times 6 = 48, \quad 12 \times 4 = 48$$

Since both products are equal, the ratios are proportional.

Practice Set

- Level 1: Check if 3 : 5 and 6 : 10 are proportional.
- Level 2: If 5 kg of sugar costs ₹150, find the cost of 8 kg.
- Level 3: A recipe uses ingredients in ratio 3 : 4 : 5. If 12 cups of the third ingredient are used, find the quantities of the other two.

Answer Key

- Level 1: Yes, proportional since 3×10 = 5×6 = 30.
- Level 2: Cost for 1 kg = 150/5 = ₹30; for 8 kg = 8×30 = ₹240.
- Level 3: Ratio parts sum = 3 + 4 + 5 = 12; 5 parts = 12 cups, so 1 part = 12/5 = 2.4 cups; 3 parts = 3×2.4 = 7.2 cups; 4 parts = 4×2.4 = 9.6 cups.

Quick Reference

Two ratios $a : b$ and $c : d$ are proportional if $a \times d = b \times c$.

Glossary

- **Ratio:** A comparison of two quantities.
- **Proportionality:** A relationship where two ratios are equal.
- **Cross multiplication:** Multiplying the outer terms and inner terms of two ratios to check equality.

Ratios in Maps

Maps use ratios called Representative Fractions (RF) to represent the scale between distances on the map and actual distances on the ground.

Concept Explanation

An RF of 1 : 60,00,000 means 1 cm on the map corresponds to 60,00,000 cm (or 60 km) in reality.

Worked Illustration

If the distance between two cities on the map is 5 cm, the actual distance is:

$$5 \text{ cm} \times 60,00,000 = 3,00,00,000 \text{ cm} = 3,000 \text{ km}$$

Solved Example

Using a map with scale 1 : 50,00,000, find the actual distance between two points 8 cm apart on the map.

Actual distance = 8 cm × 50,00,000 = 4,00,00,000 cm = 4,000 km.

Practice Set

- Level 1: Convert 1 : 1,00,000 scale to km per cm.
- Level 2: On a map with scale 1 : 25,00,000, find the real distance for 12 cm on the map.
- Level 3: If two cities are 15 cm apart on a map with scale 1 : 10,00,000, find the distance in km.

Answer Key

- Level 1: 1 cm = 1,00,000 cm = 1 km.
- Level 2: 12 cm × 25,00,000 = 3,00,00,000 cm = 3,000 km.
- Level 3: 15 cm × 10,00,000 = 1,50,00,000 cm = 1,500 km.

Quick Reference

Representative Fraction (RF) = Map distance : Actual distance

Glossary

- **Representative Fraction (RF):** Ratio of map distance to actual ground distance.

- **Scale:** The ratio that relates map measurements to real-world measurements.

Ratios with More Than Two Terms

Ratios can compare more than two quantities simultaneously, such as $a : b : c : d$.

Formula Derivation

Two ratios with multiple terms $a : b : c : d :: p : q : r : s$ are proportional if:

$$\frac{a}{p} = \frac{b}{q} = \frac{c}{r} = \frac{d}{s}$$

Worked Illustration

Viswanath's spice mix ratio: 8 : 4 : 2 : 1

Puneet has 2 red chillies (half of 4), so all quantities are halved:

4 : 2 : 1 : 0.5

Solved Example

Paint ratio Red : Blue : White = 2 : 3 : 5. If 10 litres of white paint is used, find litres of red and blue paint.

1 part = 10 ÷ 5 = 2 litres

Red = 2 × 2 = 4 litres

Blue = 3 × 2 = 6 litres

Practice Set

- Level 1: Simplify the ratio 12 : 18 : 24.
- Level 2: If cement, sand, and gravel are mixed in ratio 1 : 1.5 : 3, find quantities needed for 6 bags of cement.
- Level 3: A mixture has ratio 5 : 7 : 8 : 10. If the quantity of the last component is 40 units, find the quantities of the others.

Answer Key

- Level 1: Simplest form is 2 : 3 : 4.
- Level 2: Sand = 1.5 × 6 = 9 bags; Gravel = 3 × 6 = 18 bags.
- Level 3: 1 part = 40 ÷ 10 = 4 units; Quantities: 20, 28, 32, 40 units.

Quick Reference

For ratios $a : b : c : d$, proportionality means $\frac{a}{p} = \frac{b}{q} = \frac{c}{r} = \frac{d}{s}$.

Glossary

- **Multiple-term ratio:** Ratio involving more than two quantities.
- **Proportional ratios:** Ratios where corresponding terms have equal quotients.

Dividing a Whole in a Given Ratio

To divide a quantity x in ratio $a : b : c : \ldots$, sum the parts and multiply each term by $\frac{x}{a+b+c+\ldots}$.

Formula Derivation

$$\text{Part}_1 = x \times \frac{a}{a+b+c+\ldots}, \quad \text{Part}_2 = x \times \frac{b}{a+b+c+\ldots}, \quad \ldots$$

Worked Illustration

Divide 110 units of concrete in ratio 1 : 1.5 : 3.

Sum = 1 + 1.5 + 3 = 5.5

Multiplier = 110 ÷ 5.5 = 20

Cement = 1 × 20 = 20 units

Sand = 1.5 × 20 = 30 units

Gravel = 3 × 20 = 60 units

Solved Example

Divide 50 ml of paint in ratio 2 : 3 : 5.

Sum = 10

Red = 50 × 2/10 = 10 ml

Blue = 50 × 3/10 = 15 ml

White = 50 × 5/10 = 25 ml

Practice Set

- Level 1: Divide 24 in ratio 3 : 1.
- Level 2: Divide 100 units in ratio 4 : 5 : 6.
- Level 3: Divide 180 in ratio 2.5 : 3.5 : 4.

Answer Key

- Level 1: Sum = 4; Parts = 18 and 6.
- Level 2: Sum = 15; Parts = 26.67, 33.33, 40 units.
- Level 3: Sum = 10; Parts = 45, 63, 72 units.

Quick Reference

Divide x in ratio $a : b : c : \ldots$ by multiplying each term by $\frac{x}{a+b+c+\ldots}$.

Glossary

- **Ratio division:** Splitting a quantity into parts proportional to given ratio terms.

Inverse Proportions

Two quantities x and y are inversely proportional if their product is constant:

$$xy = k$$

When one quantity increases by a factor, the other decreases by the reciprocal factor.

Formula Derivation

If x_1, y_1 and x_2, y_2 are pairs of values, then:

$$x_1 y_1 = x_2 y_2 = k$$

Also,

$$\frac{x_1}{x_2} = \frac{y_2}{y_1}$$

Worked Illustration

Speed and time for a trip:

- Speed 5 km/h, time 18 h
- Speed 15 km/h, time 6 h
- Speed 30 km/h, time 3 h
- Speed 60 km/h, time 1.5 h

Product of speed and time is constant (90 km).

Solved Example

20 workers take 4 days to complete a job. How many days will 10 workers take?

Since workers and days are inversely proportional:

$$20 \times 4 = 10 \times x \implies x = \frac{20 \times 4}{10} = 8$$

It will take 8 days.

Practice Set

- Level 1: If 3 taps fill a tank in 6 hours, how long will 6 taps take?
- Level 2: A car takes 2 hours at 60 km/h. How long at 80 km/h?
- Level 3: Two pumps fill a tank in 18 hours. How long with 4 pumps?

Answer Key

- Level 1: 3×6 = 6×x → x = 3 hours.
- Level 2: 60×2 = 80×x → x = 1.5 hours.
- Level 3: 2×18 = 4×x → x = 9 hours.

Quick Reference

For inverse proportion, $xy = k$ constant.

Glossary

- **Inverse proportion:** One quantity increases while the other decreases such that their product is constant.