Fractals

Fractals are self-similar geometric shapes that exhibit the same or similar pattern repeatedly at smaller scales. They are found both in nature and mathematical constructs. Examples include ferns, trees, clouds, and coastlines.

Concept Explanation

Fractals are generated by iterative geometric processes where a shape is subdivided and parts are removed or replaced to create complex patterns.

Formula Derivation

Consider the Sierpinski Carpet fractal:

- Start with a square (Step 0).
- Divide it into 9 equal smaller squares and remove the central square (Step 1).
- Repeat the process on each remaining square indefinitely.

Let R_n be the number of remaining squares at step n, and H_n be the number of holes at step n.

Since each remaining square at step n produces 8 remaining squares at step $n+1$,

$$R_{n+1} = 8R_n$$

With initial condition $R_0 = 1$, the formula is:

$$R_n = 8^n$$

For holes, each remaining square at step n creates a hole at step $n+1$, and all previous holes remain:

$$H_{n+1} = H_n + R_n$$

With $H_0 = 0$, the number of holes at step n is:

$$H_n = 1 + 8 + 8^2 + \cdots + 8^{n-1} = \frac{8^n - 1}{8 - 1} = \frac{8^n - 1}{7}$$

Worked Illustrations

For the Sierpinski Carpet:

- Step 0: 1 square, 0 holes.
- Step 1: 8 squares, 1 hole.
- Step 2: 64 squares, 9 holes.

Sierpinski Gasket

Constructed by dividing an equilateral triangle into 4 smaller equilateral triangles by joining midpoints and removing the central triangle, then repeating the process on the remaining triangles.

Worked Illustration

- Step 0: 1 triangle.
- Step 1: 3 triangles (after removing the central one).
- Step 2: 9 triangles.

Koch Snowflake

Start with an equilateral triangle. Each side is divided into three equal parts, and an equilateral triangle is constructed on the middle segment, then the middle segment is removed. Repeat this process on every side indefinitely.

Worked Illustration

- Step 0: 3 sides.
- Step 1: 12 sides.
- Step 2: 48 sides.

Practice Set

- Draw the first three steps of the Sierpinski Carpet and Sierpinski Gasket.

- Calculate R_n and H_n for $n = 3$ in the Sierpinski Carpet.
- Find the number of sides and perimeter of the Koch Snowflake at step $n = 3$ given the initial side length 1 unit.
- Explore fractal patterns in art and nature and identify self-similarity.

Answer Key

- $R_3 = 8^3 = 512$, $H_3 = \frac{8^3-1}{7} = 73$
- Koch Snowflake sides at step 3: $3 \times 4^3 = 192$
- Perimeter at step n: $P_n = 3 \times (\frac{4}{3})^n$

Quick Reference

Fractal	Initial Shape	Iteration Rule	Formula
Sierpinski Carpet	Square	Remove center square of 9	$R_n = 8^n$
Sierpinski Gasket	Equilateral Triangle	Remove central triangle	Number of triangles at step n: 3^n
Koch Snowflake	Equilateral Triangle	Add smaller triangle on middle third	Number of sides at step n: 3×4^n

Glossary

- **Fractal:** A shape exhibiting self-similarity at different scales.
- **Self-similarity:** Property of a shape to look similar at various scales.
- **Sierpinski Carpet:** A fractal formed by removing central squares iteratively.
- **Sierpinski Gasket:** A fractal formed by removing central triangles iteratively.
- **Koch Snowflake:** A fractal curve with infinite perimeter but finite area.

Visualising Solids

Visualising solids involves understanding their shapes, profiles, and representations from different viewpoints without necessarily drawing them.

Concept Explanation

Profiles of solids change with viewpoint. For example, a cube may appear as a square, rectangle, or hexagon depending on orientation.

Worked Illustrations

- Visualise cutting corners of squares and triangles and observe resulting shapes.
- Identify solids with given profiles from different viewpoints.
- Visualise solids with contrasting profiles from different directions.

Practice Set

- Describe solids with square, circular, triangular profiles.
- Visualise solids with rectangular and circular profiles from different views.
- Draw solids with given contrasting profiles.

Answer Key

- Cube has square profile from front, hexagonal in isometric view.
- Cylinder has circular profile from top and rectangular from side.
- Triangular prism has triangular and rectangular profiles.

Quick Reference

- **Profile:** Outline of a solid from a viewpoint.
- **Viewpoint:** Direction from which a solid is observed.

Glossary

- **Profile:** The 2D outline of a 3D object from a specific viewpoint.
- **Viewpoint:** The position or angle from which an object is observed.

Nets of Solids

A net is a two-dimensional shape that can be folded to form a three-dimensional solid.

Concept Explanation

Nets are created by unfolding the faces of a solid onto a plane. Different solids have characteristic nets.

Worked Illustrations

- Cube net: 6 connected squares arranged to fold into a cube.
- Regular tetrahedron net: 4 equilateral triangles arranged appropriately.
- Prisms and pyramids nets based on their polygonal bases.
- Cylinder net: rectangle (side) and two circles (bases).
- Cone net: sector of a circle (lateral surface) and a circle (base).

Practice Set

- Draw nets for cube, cuboid, tetrahedron, square pyramid, cylinder, cone, triangular prism.
- Identify valid nets for given solids.
- Make physical models from nets.

Answer Key

- Cube has 11 distinct nets.
- Regular tetrahedron has 2 nets.
- Cylinder net consists of a rectangle and two circles.
- Cone net is a sector of a circle plus a circle.

Quick Reference

- **Net:** 2D layout of faces of a solid.
- **Foldable surface:** Material used to create nets.

Glossary

- **Net:** A flat pattern that can be folded to form a 3D solid.
- **Face:** A flat surface of a solid.
- **Edge:** Line segment where two faces meet.
- **Vertex:** Point where edges meet.

Shortest Paths on a Cuboid

Finding the shortest path between two points on the surface of a cuboid involves unfolding the cuboid into nets and finding straight line distances on the nets.

Concept Explanation

The shortest path on the surface corresponds to a straight line on some net of the cuboid.

Worked Illustrations

- Unfold cuboid into nets in different ways.
- Draw straight lines between points on nets.

- Calculate distances using Pythagoras theorem.

Formula Derivation

For a cuboid with dimensions l, w, h, shortest path between two points on adjacent faces can be found by flattening the faces and applying the Pythagorean theorem.

Practice Set

- Find shortest path for given points on cuboid surfaces.
- Explore different nets to find minimal distances.
- Apply Pythagorean theorem to calculate distances.

Answer Key

- Calculate distances on nets and choose minimum.
- Example: Distance $d = \sqrt{24^2 + 32^2} = 40$ cm.

Quick Reference

- **Shortest path:** Minimum distance along the surface between two points.
- **Net:** Unfolded faces of the cuboid.

Glossary

- **Net:** Flattened layout of a solid's faces.
- **Pythagorean theorem:** $a^2 + b^2 = c^2$ for right triangles.

Projections of Solids

Projections represent a 3D object on a 2D plane by mapping points perpendicularly onto the plane.

Concept Explanation

Projection of a point P onto a plane M is the foot of the perpendicular from P to M.

Projections of all points of an object form its projection on the plane.

Worked Illustrations

- Projection of lines and polygons.
- Projections of solids like cubes and cones.
- Front, top, and side views correspond to projections on vertical, horizontal, and side planes.

Practice Set

- Draw projections of various solids from different viewpoints.
- Identify solids from given projections.
- Compare lengths of projected lines with actual lengths.

Answer Key

- Projection length p is less than or equal to actual length l.
- Projection of a square remains a parallelogram under projection.

Quick Reference

- **Projection:** Mapping of points onto a plane via perpendicular lines.
- **Front view:** Projection on vertical plane.
- **Top view:** Projection on horizontal plane.
- **Side view:** Projection on side plane.

Glossary

- **Projection:** The image of a point or object on a plane formed by perpendicular lines.
- **Vertical plane:** Plane standing upright in front of the object.
- **Horizontal plane:** Plane lying flat below the object.
- **Side plane:** Plane to the side of the object.

Isometric Projections

Isometric projection is a method of visually representing three-dimensional objects in two dimensions, where the lengths of edges along the three principal axes are equal.

Concept Explanation

By orienting a cube so that it rests on a vertex, the projection onto a plane shows all edges equally foreshortened, forming a regular hexagon.

Worked Illustrations

- Drawing cubes and solids on isometric grid paper.
- Representing Tetris shapes as arrangements of cubes on isometric grids.
- Understanding correspondence between grid directions and solid axes.

Practice Set

- Draw cubes and other solids on isometric grid paper.

- Draw Tetris shapes as 3D solids on isometric grids.
- Visualise and draw complex shapes using isometric projection.

Answer Key

- Isometric projection preserves equal edge lengths along three axes.
- Edges project to three directions: vertical (height), and two diagonal directions (length and depth).

Quick Reference

- **Isometric projection:** Equal foreshortening of edges along three axes.
- **Isometric grid:** Triangular grid used for drawing isometric projections.

Glossary

- **Isometric:** Equal measure; equal scale along axes.
- **Isometric grid:** Grid of equilateral triangles used for isometric drawing.