Algebra Play

Algebra allows us to model and analyze numerical puzzles and tricks systematically. We use letter-numbers (variables) to represent unknowns and derive formulas that explain why certain tricks always yield the same result. This section explores various algebraic puzzles, including "Think of a Number" tricks, number pyramids, algebraic grids, and divisibility properties.

Formula Derivation and Explanation

Consider the classic "Think of a Number" trick:

1. Think of a number: x
2. Double it: $2x$
3. Add four: $2x + 4$
4. Divide by two: $\frac{2x+4}{2} = x + 2$
5. Subtract the original number: $(x + 2) - x = 2$

This shows the final result is always 2, regardless of the starting number.

Worked Illustration

Try with $x = 5$:

1. Double: $2 \times 5 = 10$
2. Add 4: $10 + 4 = 14$
3. Divide by 2: $14/2 = 7$
4. Subtract original: $7 - 5 = 2$

The result is 2, confirming the formula.

Solved Example

Modify the trick to get a final answer of 3:

1. Think of a number: x
2. Double it: $2x$
3. Add k: $2x + k$
4. Divide by 2: $x + \frac{k}{2}$
5. Subtract x: $\frac{k}{2}$

Set $\frac{k}{2} = 3$ so $k = 6$. Thus, adding 6 instead of 4 will yield 3.

Practice Set

- Level 1: Try the trick with different values of k to get final answers 4, 5, and 10.
- Level 2: Create a trick with three operations that always results in 7.
- Level 3: Invent a multi-step trick involving multiplication, addition, and division that always yields 12.

Answer Key

- Level 1: $k = 8, 10, 20$ respectively.
- Level 2 and 3: Answers vary; verify by algebraic derivation.

Quick Reference

Final result after operations $=$ constant $=$ expression independent of x.

Glossary

- **Variable:** A symbol representing an unknown number.
- **Expression:** A combination of variables and numbers using operations.
- **Equation:** A statement that two expressions are equal.

Number Pyramids

Number pyramids are arrangements where each number above is the sum of the two numbers directly below it. They help visualize addition and algebraic relationships.

Formula Derivation

For a three-layer pyramid with bottom row a, b, c:

- Middle row: $a + b, b + c$
- Top: $(a + b) + (b + c) = a + 2b + c$

Worked Illustration

Given bottom row: 1, 9, 4

1. Middle row: 1 + 9 = 10, 9 + 4 = 13
2. Top: 10 + 13 = 23

Solved Example

Find a, b, c if top is 60, bottom row is 12, c, 8, and middle row is a, b:

1. $a + b = 60$
2. $12 + c = a$
3. $c + 8 = b$
4. Substitute: $(12 + c) + (c + 8) = 60 \Rightarrow 20 + 2c = 60 \Rightarrow c = 20$
5. Then $a = 12 + 20 = 32$, $b = 20 + 8 = 28$

Practice Set

- Level 1: Complete pyramids with given bottom rows.
- Level 2: Express top number in terms of bottom row variables for 4-layer pyramids.
- Level 3: Explore pyramids with Fibonacci numbers as bottom rows and analyze the top number.

Answer Key

- Level 1: Calculated by successive addition.
- Level 2: Sum of bottom row variables weighted by binomial coefficients.
- Level 3: Top number relates to sums of Fibonacci numbers.

Quick Reference

Top number in 3-layer pyramid: $a + 2b + c$

Glossary

- **Number Pyramid:** A triangular arrangement where each number is the sum of two below.
- **Fibonacci Sequence:** A sequence where each number is the sum of the two preceding ones.

Algebra Grids

Algebra grids use shapes to represent unknown numbers. The sums of rows or columns provide equations to solve for these unknowns.

Formula Derivation

Example: Three blue squares sum to 27:

$$3 \times \text{blue square} = 27 \Rightarrow \text{blue square} = 9$$

Two red circles and one blue square sum to 19:

$$2 \times \text{red circle} + 9 = 19 \Rightarrow \text{red circle} = 5$$

Worked Illustration

Given grid with shapes and sums, write equations for each row and column, then solve simultaneously.

Solved Example

Find values of shapes in grids with given row and column sums by setting variables and solving linear equations.

Practice Set

- Level 1: Solve grids with two shapes and given sums.
- Level 2: Solve grids with three shapes and multiple sums.
- Level 3: Create your own algebra grid puzzles.

Answer Key

Solutions depend on the system of equations formed; verify by substitution.

Quick Reference

Sum of shapes in a row or column = given total; use algebra to solve.

Glossary

- **Variable:** Unknown value represented by a shape.
- **Equation:** Mathematical statement equating two expressions.

Largest Product Problem

Given digits, arrange them to form a two-digit number multiplied by a one-digit number to maximize the product.

Formula Derivation

Digits $p < q < r$. Possible products:

- $qr \times p$
- $rq \times p$
- $pr \times q$
- $rp \times q$
- $pq \times r$
- $qp \times r$

Compare pairs with same multiplier; largest product is $qp \times r$ where r is largest digit.

Worked Illustration

Digits 2, 3, 5:

- Compare 53 × 2, 52 × 3, 32 × 5
- Calculate: 53 × 2 = 106, 52 × 3 = 156, 32 × 5 = 160
- Largest product is 160 from 32 × 5

Solved Example

Digits 1, 3, 7:

- Possible largest product: 31 × 7 = 217
- Check others to confirm

Practice Set

- Level 1: Digits 1, 3, 7
- Level 2: Digits 3, 5, 9

- Level 3: Digits 4, 6, 8

Answer Key

- Level 1: 31 × 7 = 217
- Level 2: 53 × 9 = 477
- Level 3: 64 × 8 = 512

Quick Reference

Use largest digit as multiplier; arrange other two digits in decreasing order for multiplicand.

Glossary

- **Product:** Result of multiplication.
- **Multiplicand:** Number being multiplied.
- **Multiplier:** Number by which multiplicand is multiplied.

Divisibility Tricks

Explore properties of numbers related to digit reversal and divisibility by 9, 11, 37, and others.

Formula Derivation

Two-digit number $ab = 10a + b$, reversed $ba = 10b + a$.

Difference:

$$(10b + a) - (10a + b) = 9(b - a)$$

Always divisible by 9.

Worked Illustration

Number 47 and 74:

- Difference: 74 - 47 = 27
- 27 ÷ 9 = 3, no remainder

Solved Example

Sum of number and its reverse is divisible by 11:

$$(10a + b) + (10b + a) = 11(a + b)$$

Practice Set

- Level 1: Verify divisibility by 9 for various two-digit numbers.
- Level 2: Prove sum of two-digit number and its reverse divisible by 11.
- Level 3: For three-digit number abc, prove sum of cyclic permutations divisible by 37.

Answer Key

- Level 1: Difference always multiple of 9.

- Level 2: Sum always multiple of 11.
- Level 3: Sum of $abc + bca + cab = 111(a+b+c)$, divisible by 37.

Quick Reference

Difference of reversed two-digit numbers divisible by 9; sum divisible by 11.

Glossary

- **Divisible:** A number is divisible by another if division leaves no remainder.
- **Digit reversal:** Writing digits of a number in reverse order.