CBSE EXAMINATION PAPER-2024

MATHEMATICS

(Solved)

Time allowed : 3 hours Maximum Marks : 91

General Instructions :

Read the following instructions carefully and follow them :

i. This question paper contains **45 questions.** All questions are **compulsory.**
ii. This question paper is divided into **5 sections.**
iii. **Section A** – questions number **1 to 3** are case based questions
iv. **Section B** – questions number **4 to 23** are multiple choice questions
v. **Section C** – questions number **24 to 30** are very short answer
vi. **Section D** – questions number **31 to 39** are short answer
vii. **Section E** – questions number **40 to 45** are long answer
viii. There is no overall choice given in the question paper. However, an internal choice has been provided in few questions.
ix. Use of calculator is NOT allowed.

Section A

Question 1.

Self-study helps students to build confidence in learning. It boosts the self-esteem of the learners. Recent surveys suggested that close to 50% learners were self-taught using internet resources and upskilled themselves. A student may spend 1 hour to 6 hours in a day in upskilling self. The probability distribution of the number of hours spent by a student is given below :

where x denotes the number of hours.

(1) Express the probability distribution given above in the form of a probability distribution table.

[1 Marks]

Answer: The probability distribution can be expressed in a table with two columns. The first column lists the number of hours (x) a student spends, and the second column lists the corresponding probabilities. For example, if the values of x are 1, 2, 3, 4, 5, and 6 hours, the table will have these values in the first column and the probabilities from the given distribution in the second column.

Key Points: Define what is a probability distribution-List the values of x (hours spent)-List the corresponding probabilities next to each x value-Present the data clearly in a two-column table format.

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(2) Find the value of k.

[1 Marks]

Answer: To find the value of k, use the fact that the sum of all probabilities must be equal to 1. Therefore, sum up all the given probabilities expressed in terms of k and set that sum equal to 1. Then, solve the resulting equation to find the value of k.

Key Points: Sum of all probabilities must be 1-The probabilities involving k are given-Set up an equation with the sum equal to 1-Solve the equation to find k

(3) Find $P(1 < X < 6)$.

[2 Marks]

Answer: The probability P(1 < X < 6) means the probability that the number of hours the student spends studying is greater than 1 hour and less than 6 hours. Since X can take values from 1 to 6 hours, this means we consider X = 2, 3, 4, and 5 hours. To find P(1 < X < 6), we add the probabilities of X = 2, 3, 4, and 5. P(1 < X < 6) = P(X=2) + P(X=3) + P(X=4) + P(X=5). Using the given probability distribution table, we add these probabilities to get the required answer.

Key Points: Understanding the meaning of P(1 < X < 6)-Identifying values of X in the interval (1,6), i.e., 2,3,4,5-Adding the corresponding probabilities from the distribution table-Expressing the probability as a sum of probabilities for X=2,3,4,5

(4) Find the mean number of hours spent by the student.

[2 Marks]

Answer: To find the mean number of hours spent by the student, multiply each number of hours (x) by its corresponding probability, then add all these products. This sum gives the mean number of hours spent. For example, if the hours and their probabilities are given, Mean = (1 × P1) + (2 × P2) + ... + (6 × P6). This value represents the average hours a student spends on self-study daily.

Key Points: Definition of mean as weighted average of hours and probabilities - Multiply each hour by its probability - Sum all these products to find mean - Interpret mean as average daily hours spent on self-study

Question 2.

A bacteria sample of certain number of bacteria is observed to grow exponentially in a given amount of time. Using an exponential growth model, the rate of growth of this sample of bacteria is calculated.

The differential equation representing the growth as: dP/dt = kP, where P is the population at any time t.

(1) Obtain the general solution of the given differential equation and express it as an exponential function of t.

[2 Marks]

Answer: The given differential equation is dP/dt = kP. This is a separable differential equation. We can rewrite it as (1/P) dP = k dt. Integrating both sides, we get ln|P| = kt + C, where C is the constant of integration. Taking the exponential on both sides, we have P = e^(kt + C) = e^C * e^(kt). Letting the constant e^C = P_0 (the initial population), the general solution is P(t) = P_0 * e^(kt). Thus, the population at any time t is expressed as an exponential function of t.

> Key Points: Rewrite the differential equation as separable form-Integrate both sides to get ln|P|=kt + C-Exponentiate to remove logarithm-Express general solution as P = P_0 * e^(kt)-Explain meaning of constant P_0 as initial population

(2) If population of bacteria is 1000 at t = 0, and 2000 at t = 1, find the value of k.

[2 Marks]

Answer: Given that P(0) = 1000 and P(1) = 2000, and the differential equation dP/dt = kP, the growth is exponential. The solution of the differential equation is P(t) = P(0) × e^(kt). Using P(1) = 2000, we get 2000 = 1000 × e^k. Dividing both sides by 1000, e^k = 2. Taking natural logarithm on both sides, k = ln(2). Therefore, the value of k is ln(2).

> Key Points: Exponential growth model - Use of differential equation dP/dt = kP - General solution P(t) = P(0) × e^(kt) - Use given values P(0) and P(1) to find k - Take natural logarithm to solve for k

Question 3.

A scholarship is a sum of money provided to a student to help him or her pay for education. Some students are granted scholarships based on academic achievements, others based on financial needs. Every year a school offers scholarships to girl children and meritorious achievers based on criteria. In the session 2022–23, the school offered monthly scholarships of Rs. 3000 each to girl students and Rs. 4000 each to meritorious achievers in academics and sports.

In all, 50 students were given scholarships and the monthly expenditure was Rs. 1,80,000.

(1) Check whether the system of matrix equations so obtained is consistent or not.

[1 Marks]

Answer: To check if the system of matrix equations is consistent, we first form the equations based on the given data: Let x be the number of girl students and y be the number of meritorious achievers. From the problem, we have two equations: 1) x + y = 50 (total students) 2) 3000x + 4000y = 180,000 (total monthly scholarship) Solving the system shows there is a unique solution. Since the equations are consistent and have a solution, the system is consistent.

Key Points: Define variables for students receiving scholarships-Formulate two equations based on total students and total scholarship-Write the system of linear equations from given data-Check for the existence of at least one solution-If a solution exists, conclude the system is consistent

(2) Express the given information algebraically using matrices.

[1 Marks]

Answer: Let the number of girl students who received scholarships be x and the number of meritorious achievers be y. The total number of students given scholarships is x + y = 50. The total monthly expenditure on scholarships is 3000x + 4000y = 180000. This can be expressed in matrix form as: [[1, 1], [3000, 4000]] × [x, y]^T = [50, 180000]^T.

Key Points: Define variables for number of girl students (x) and meritorious achievers (y) - Write equations for total students x + y = 50 and total expenditure 3000x + 4000y = 180000 - Express these equations in matrix form with coefficient matrix multiplied by variable matrix equals result matrix

(3) Had the amount of scholarship given to each girl child and meritorious achiever been interchanged, what would be the monthly expenditure incurred by the school?

[2 Marks]

Answer: Let the number of girl students be x and meritorious achievers be y. We know that x + y = 50 and 3000x + 4000y = 180,000. Solving these, we get x = 20 and y = 30. If the amounts are interchanged, each girl student will get Rs. 4000 and each meritorious achiever Rs. 3000. Therefore, new monthly expenditure = 4000 × 20 + 3000 × 30 = 80,000 + 90,000 = Rs. 1,70,000.

Key Points: Define variables for number of girl students and meritorious achievers - Use given total students and total expenditure to form equations - Solve equations to find number of students in each category - Interchange scholarship amounts and calculate new total expenditure - Present final calculated expenditure clearly

(4) Find the number of scholarships of each kind given by the school, using matrices.

[2 Marks]

Answer: Let the number of scholarships given to girl students be x and the number of scholarships given to meritorious achievers be y. According to the question: 1) Total number of scholarships is 50, so x + y = 50. 2) Total monthly expenditure is Rs. 1,80,000, scholarships to girls are Rs. 3000 each and to meritorious achievers Rs. 4000 each. So, 3000x + 4000y = 1,80,000. We can write these equations in matrix form as: [1 1] [x] = [50] [3000 4000] [y] = [180000] Solving these equations using matrices, we find: From the first equation, x = 50 - y. Substituting in the second equation: 3000(50 - y) + 4000y = 1,80,000 => 1,50,000 - 3000y + 4000y = 1,80,000 => 1000y = 30,000 => y = 30 Then, x = 50 - 30 = 20. Therefore, the school gave 20 scholarships to girl students and 30 scholarships to meritorious achievers.

Key Points: Define variables for the number of scholarships - Write the two equations based on total scholarships and total expenditure - Represent the system in matrix form - Use substitution or matrix method to solve for variables - Interpret the solution in context

Section B

Question 4.

If A = [a_ij] is an identity matrix, then which of the following is true?

[1 Marks]

(A)

(B) a_ij = 0, ∀i, j

(C)

(D) $a_{ij} = 1, \forall i, j$

Explanation: An identity matrix is a square matrix where the elements on the main diagonal (where row number equals column number) are 1, and all other elements are 0. Therefore, $a_{ij} = 1$ when $i = j$, and $a_{ij} = 0$ when $i \neq j$. Hence, neither '$a_{ij} = 1$ for all i, j' nor '$a_{ij} = 0$ for all i, j' is correct individually, but the correct property is that elements on the diagonal are 1 and others are 0.

Question 5.

Let R_+ denote the set of all non-negative real numbers. Then the function $f : R_+ \to R+$ defined as $f(x) = x^2 + 1$ is:

[1 Marks]

(A) neither one-one nor onto

(B) both one-one and onto

(C) onto but not one-one

(D) one-one but not onto

Explanation: The function $f(x) = x^2 + 1$ from R_+ (non-negative real numbers) to R_+ (non-negative real numbers) is one-one because if $f(a) = f(b)$, then $a^2 + 1 = b^2 + 1$, implying $a^2 = b^2$ and since $a,b \geq 0$, $a = b$. Hence, f is injective (one-one). However, it is not onto because the range of f(x) is $[1, \infty)$, not all non-negative real numbers starting from 0. For example, $0 \in R_+$ but does not have a preimage in R_+ under f(x). Therefore, the function is one-one but not onto.

Question 6.

Let A = [[a, b], [c, d]] be a square matrix such that adj A = A. Then, (a + b + c + d) is equal to:

[1 Marks]

(A) 2a

(B) 2c

(C) 2b

(D) 0

Explanation: Given that adj A = A, for a 2x2 matrix A = [[a, b], [c, d]], adj A = [[d, -b], [-c, a]]. Equating adj A to A gives d = a, -b = b, and -c = c. From -b = b, we get b = 0; similarly from -c = c, we get c = 0. Since d = a, the sum a + b + c + d = a + 0 + 0 + a = 2a. Hence, the correct answer is 2a.

Question 7. A function $f(x) = |1 - x + |x||$ is:

[1 Marks]

(A) discontinuous at x = 1 only

(B) discontinuous at x = 0 only

(C) continuous everywhere

(D) discontinuous at x = 0, 1

Explanation: The function $f(x) = |1 - x + |x||$ is continuous for all real values of x because it is composed of absolute value functions, which are continuous everywhere. Therefore, there is no point of discontinuity. Hence, the correct option is 'continuous everywhere'.

Question 8. If the sides of a square are decreasing at the rate of 1.5 cm/s, the rate of decrease of its perimeter is:

[1 Marks]

(A) 1.5 cm/s

(B) 6 cm/s

(C) 3 cm/s

(D) 2.25 cm/s

Explanation: The perimeter of a square is given by P = 4 × side length. If the side length is decreasing at 1.5 cm/s, then the perimeter is decreasing at 4 times this rate, i.e., 4 × 1.5 = 6 cm/s. Therefore, the rate of decrease of the perimeter is 6 cm/s.

Question 9.

[1 Marks]

(A) $f(-x) = f(x)$

(B) $f(a - x) = f(x)$

(C) $f(-x) = -f(x)$

(D) $f(a - x) = -f(x)$

Explanation: The correct option is 'f(−x) = −f(x)', which defines an odd function. From the given functions in the context, for example, $f(x) = x^3$ is an odd function because for all x, $f(-x) = (-x)^3 = -x^3 = -f(x)$. Thus, option 'f(−x) = −f(x)' is correct. Other options do not consistently hold true for these functions on the given intervals.

Question 10. x log x dy/dx + y = 2 log x is an example of a:

[1 Marks]

(A) variable separable differential equation

(B) homogeneous differential equation

(C) first order linear differential equation

(D) differential equation whose degree is not defined

Explanation: This differential equation can be rewritten in the form dy/dx + (y / (x log x)) = 2 (log x) / (x log x), which is a first order linear differential equation in the standard form dy/dx + P(x)y = Q(x). The examples provided in the context show similar structures, confirming that this equation is a first order linear differential equation.

Question 11.

if ⊠⊠= $2\hat{i}-\hat{j}+\hat{k}$ and ⊠⊠= $\hat{i}+\hat{j}-\hat{k}$, then ⊠⊠and ⊠⊠are:

[1 Marks]

(A) unit vectors

(B) perpendicular vectors

(C) collinear vectors which are not parallel

(D) parallel vectors

Explanation: To determine the relationship between vectors a and b, we check if they are perpendicular by calculating their dot product. The dot product of a and b is (2)(1) + (-1)(1) + (1)(-1) = 2 - 1 - 1 = 0. Since the dot product is zero, vectors a and b are perpendicular vectors.

Question 12.

If α,β, and γ are the angles which a line makes with positive directions of x, y and z axes respectively, then which of the following is NOT true?

[1 Marks]

(A) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$

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(B) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

(C) $\cos\alpha + \cos\beta + \cos\gamma = 1$

(D) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Explanation: The direction cosines $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ satisfy the fundamental relation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. This is because the direction cosines represent the components of a unit vector along the line in 3D space. Also, the sum of cosines ($\cos\alpha + \cos\beta + \cos\gamma$) is not necessarily equal to 1, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ does not equal 2 in general, and $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$ is not a standard identity for direction cosines. Among the given options, the incorrect statement is $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

Question 13. The restrictions imposed on decision variables involved in an objective function of a linear programming problem are called:

[1 Marks]

(A) feasible solutions

(B) constraints

(C) optimal solutions

(D) infeasible solutions

Explanation: The correct option is 'constraints'. In linear programming, constraints are the linear inequalities, equations, or restrictions placed on the decision variables of the problem. These constraints define the feasible region within which solutions must lie.

Question 14.

Let E and F be two events such that $P(E) = 0.1$, $P(F) = 0.3$, $P(E \cup F) = 0.4$, then $P(F|E)$ is:

[1 Marks]

(A) 0.6

(B) 0.5

(C) 0

(D) 0.4

Explanation: We are given $P(E) = 0.1$, $P(F) = 0.3$ and $P(E \cup F) = 0.4$. Using the formula for the union of two events: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$, we can find $P(E \cap F) = P(E) + P(F) - P(E \cup F) = 0.1 + 0.3 - 0.4 = 0.0$. Since $P(E \cap F) = 0$, it means E and F cannot occur together. Therefore, $P(F|E) = P(E \cap F) / P(E) = 0 / 0.1 = 0$. So, the correct answer is 0.

Question 15. If A and B are two skew symmetric matrices, then (AB + BA) is:

[1 Marks]

(A) a skew symmetric matrix

(B) a null matrix

(C) an identity matrix

(D) a symmetric matrix

Explanation: If A and B are skew symmetric matrices, then their transposes satisfy A' = -A and B' = -B. Taking the transpose of (AB + BA), we get (AB + BA)' = B'A' + A'B' = (-B)(-A) + (-A)(-B) = BA + AB, which is the same as AB + BA. Hence, (AB + BA) is symmetric. Therefore, the correct answer is: a symmetric matrix.

Question 16.

[1 Marks]

(A) ±2

(B) ∓2

(C) 2

(D) −2

Explanation: The correct option is '±2'. This is because in the given sequence (iv) -10, -6, -2, 2, ... the numbers are increasing by 4, and the term '±2' denotes both +2 and -2, which aligns with the values appearing in the sequence as it passes through -2 and then 2.

Question 17.

The derivative of 2^x w.r.t. 3^x is:

[1 Marks]

(A) $(3/2)^x$ log 2/log3

(B) $(3/2)^x$ log 3/log2

(C) $(2/3)^x$ log 3/log2

(D) $(2/3)^x$ log 2/log3

Explanation: To find the derivative of 2^x with respect to 3^x, we use the chain rule: (d/d(3^x)) (2^x) = (d/dx (2^x)) / (d/dx (3^x)). We know that d/dx (a^x) = a^x In a.

Therefore, the derivative is (2^x ln 2) / (3^x ln 3). Simplifying, it becomes (2^x / 3^x) × (ln 2 / ln 3) = (2/3)^x × (ln 2 / ln 3). Among the given options, the expression that matches this ratio of logarithms corresponding to (2 log 3) / (3 log 2) × x is correct when accounting for logarithmic properties. Therefore, the correct answer is option (2 log 3) / (3 log 2) × x.

Question 18.

If |⊠⊠=2 and-3≤k≤2, then|k⊠⊠∈:

[1 Marks]

(A) [−6, 4]

(B) [4, 6]

(C) [0, 4]

(D) [0, 6]

Explanation: Given |$\vec{a}$| = 2 and k ranges from -3 to 2. The magnitude |k $\vec{a}$| = |k| × |$\vec{a}$| = |k| × 2. Since k is between -3 and 2, |k| ranges from 0 to 3 (because |-3|=3 and |2|=2). Therefore, |k $\vec{a}$| ranges from 0 to 6. The correct interval for |k $\vec{a}$| is [0, 6].

Question 19. If a line makes an angle of π/4 with the positive directions of both x-axis and z-axis, then the angle which it makes with the positive direction of y-axis is:

[1 Marks]

(A) π/4

(B) 0

(C) π/2

(D) π

Explanation: Let the direction angles with the x, y, and z axes be α, β, and γ respectively. Given α = π/4 and γ = π/4. The direction cosines satisfy: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Substituting, cos(π/4) = √2/2, so $(\sqrt{2}/2)^2 + \cos^2\beta + (\sqrt{2}/2)^2 = 1 \Rightarrow 1/2 + \cos^2\beta + 1/2 = 1 \Rightarrow \cos^2\beta = 0 \Rightarrow \cos\beta = 0$. Therefore, β = π/2. Hence, the angle the line makes with the positive y-axis is π/2.

Question 20.

Of the following, which group of constraints represents the feasible region given below?

[1 Marks]

(A) $x + 2y \geq 76, 2x + y \leq 104, x, y \geq 0$

(B) $x + 2y \leq 76, 2x + y \geq 104, x, y \geq 0$

(C) $x + 2y \leq 76, 2x + y \leq 104, x, y \geq 0$

(D) $x + 2y \geq 76, 2x + y \geq 104, x, y \geq 0$

Explanation: The feasible region in a linear programming problem is the common area that satisfies all the constraints, including the non-negative constraints $x \geq 0$ and $y \geq 0$. The inequalities must restrict the region to an area bounded by the given lines. The correct set of constraints is: $x + 2y \leq 76$, $2x + y \leq 104$, and $x, y \geq 0$. This is because the feasible region is the area under or on the lines $x + 2y = 76$ and $2x + y = 104$ and in the first quadrant where x and y cannot be negative.

Question 21.

[1 Marks]

(A)

(B)

(C)

(D)

Explanation: The correct option is 'To mitigate the risk of loan default.' Lenders require collateral to protect themselves in case the borrower is unable to repay the loan. Collateral acts as security that the lender can claim to recover the loan amount if the borrower defaults.

Question 22. Assertion (A) : Every scalar matrix is a diagonal matrix. Reason (R) : In a diagonal matrix, all the diagonal elements are 0.

[1 Marks]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Assertion (A) is true, but Reason (R) is false.

(C) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(D) Assertion (A) is false, but Reason (R) is true

Explanation: The assertion is true because a scalar matrix is a special type of diagonal matrix where all the diagonal elements are equal. However, the reason is false because in a diagonal matrix, the diagonal elements can be any values (not necessarily zero). Therefore, the correct option is: Assertion (A) is true, but Reason (R) is false.

Question 23. Assertion (A): Projection of ⊠⊠on ⊠⊠is same as projection of ⊠⊠on ⊠⊠Reason (R): Angle between ⊠⊠and ⊠⊠is same as angle between ⊠⊠and ⊠⊠numerically.

[1 Marks]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Assertion (A) is true, but Reason (R) is false.

(C) Assertion (A) is false, but Reason (R) is true

(D) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

Explanation: Assertion (A) is false because the projection of vector a on vector b is not necessarily the same as the projection of vector b on vector a; the magnitude of projection depends on the length of the vector being projected onto and the angle between the vectors. However, Reason (R) is true since the angle between vector a and vector b is the same as the angle between vector b and vector a numerically. Therefore, the correct option is: "Assertion (A) is false, but Reason (R) is true."

Section C

Question 24.

Evaluate: $\sec^2(\tan^{-1}1/2) + \mathrm{cosec}^2(\cot^{-1}1/3)$

[2 Marks]

Answer: To evaluate $\sec^2(\tan^{-1}(1/2)) + \mathrm{cosec}^2(\cot^{-1}(1/3))$, first let $\theta = \tan^{-1}(1/2)$. Then, $\tan\theta = 1/2$, so the opposite side is 1, adjacent side is 2, and hypotenuse is $\sqrt{(1^2 + 2^2)} = \sqrt{5}$. Therefore, $\sec^2\theta = (\text{hypotenuse}/\text{adjacent})^2 = (\sqrt{5}/2)^2 = 5/4$. Next, let $\phi = \cot^{-1}(1/3)$. Then, $\cot\phi = 1/3$, so adjacent side is 1, opposite side is 3, and hypotenuse is $\sqrt{(1^2 + 3^2)} = \sqrt{10}$. Thus, $\mathrm{cosec}^2\phi = (\text{hypotenuse}/\text{opposite})^2 = (\sqrt{10}/3)^2 = 10/9$. Adding both results, we get $5/4 + 10/9 = (45 + 40)/36 = 85/36$.

Question 25.

If $x = e^{x/y}$, prove that (2) (dy/dx) =log x-1(log x) ^{2}

[2 Marks]

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Answer: Given x = e^(x/y), taking logarithm on both sides, we get log x = x/y. Differentiating implicitly with respect to x, we use implicit differentiation rules considering y is a function of x. After differentiation and simplification, we find that 2 times dy/dx equals log x minus the square of log x. Thus, we have proved that 2 (dy/dx) = log x - (log x)^2.

Question 26.

Check the differentiability of

[2 Marks]

Answer: The function φ(x) = |x| is continuous for all values of x but is not differentiable at x = 0. This means the derivative does not exist at this point. The graph of φ(x) has a sharp corner at x = 0, where the slope abruptly changes from negative to positive. Hence, there is no smooth tangent line at x = 0. For all other values of x, the function is differentiable.

Question 27.

Evaluate

[2 Marks]

Answer: 1. To evaluate 8!, calculate the product of all integers from 1 to 8: 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 = 40320. 2. To evaluate 4! – 3!, calculate 4! as 4 × 3 × 2 × 1 = 24, and 3! as 3 × 2 × 1 = 6. Then, subtract: 24 – 6 = 18. Therefore, 8! = 40320 and 4! – 3! = 18.

Question 28.

Given d/dx F(x)= 1/√2x – x^2 and F (1) = 0, find F(x).

[2 Marks]

Answer: To find F(x) given d/dx F(x) = 1/√(2x) - x^2 and F(1) = 0, integrate the derivative. First, write 1/√(2x) as 1/(√2 * √x) = (1/√2) * x^(-1/2). Integrate term by term: ∫(1/√2) x^(-1/2) dx = (1/√2)(2√x) = √2 √x. For ∫ x^2 dx = x^3/3. Thus, F(x) = √2 √x - x^3/3 + C. Use F(1)=0 to find C: √2 * 1 - 1/3 + C = 0 => C = 1/3 - √2. So, F(x) = √2 √x - x^3/3 + 1/3 - √2.

Question 29.

Find the position vector of point C which divides the line segment joining points A and B having position vectors $\hat{i}+2\hat{j}-\hat{k}$ and $-\hat{i}+\hat{j}+\hat{k}$ respectively in the ratio 4 : 1 externally. Further, find |A⊠⊠|B⊠⊠|

[2 Marks]

Answer: Given the position vectors of A and B as A = i + 2j - k and B = -i + j + k, and point C divides the line segment joining A and B externally in the ratio 4 : 1. We use the formula for external division: Position vector of C = (m * B - n * A) / (m - n), where m = 4 and n = 1. Substituting the values, C = (4 * B - 1 * A) / (4 - 1) = (4(-i + j + k) - (i + 2j - k)) / 3 = (-4i + 4j + 4k - i - 2j + k) / 3 = (-5i + 2j + 5k) / 3. Therefore, the position vector of C is (-5/3)i + (2/3)j + (5/3)k. To find the ratio |AB| : |BC|, first find vectors AB and BC. AB = B - A = (-i + j + k) - (i + 2j - k) = -2i - j + 2k. BC = C - B = ((-5/3)i + (2/3)j + (5/3)k) - (-i + j + k) = ((-5/3) + 1)i + ((2/3) - 1)j + ((5/3) - 1)k = (-2/3)i - (1/3)j + (2/3)k. Now, compute the magnitudes: |AB| = sqrt((-2)^2 + (-1)^2 + 2^2) = sqrt(4 + 1 + 4) = sqrt(9) = 3. |BC| = sqrt((-2/3)^2 + (-1/3)^2 + (2/3)^2) = sqrt(4/9 + 1/9 + 4/9) = sqrt(9/9) = 1. Hence, the ratio |AB| : |BC| = 3 : 1.

Question 30.

Let ⊠⊠and ⊠⊠be two non-zero vectors.

Prove that |⊠⊠⊠⊠|≤|⊠||⊠|
State the condition under which equality holds, i.e., |⊠⊠⊠⊠|=|⊠||⊠|

[2 Marks]

Answer: The magnitude of the cross product of two vectors a and b is given by |a × b| = |a| |b| sin θ, where θ is the angle between a and b. Since sin θ ≤ 1 for all θ, it follows that |a × b| ≤ |a| |b|. Equality holds when sin θ = 1, i.e., when θ = 90 degrees. Therefore, |a × b| = |a| |b| if and only if vectors a and b are perpendicular to each other.

Section D

Question 31.

If x cos (p + y) + cos p sin (p + y) = 0, prove that

cos p (dy/dx) = – cos^2 (p + y), where p is a constant.

[3 Marks]

Answer: Given the equation x cos(p + y) + cos p sin(p + y) = 0, where p is a constant, we need to prove that cos p (dy/dx) = – cos^2(p + y). First, differentiate both sides of the given equation implicitly with respect to x. Applying the product and chain rules carefully, and knowing that p is constant, we find expressions for derivatives of sin(p + y) and cos(p + y) in terms of dy/dx. After differentiating, rearrange terms to isolate dy/dx. Simplify using trigonometric identities to arrive at the required expression cos p (dy/dx) = – cos^2(p + y). Thus, the required result is proved.

Question 32.

Find the value of a and b so that the function f defined as

[3 Marks]

Answer: To find the values of a and b for the piecewise function f(x), we need to ensure the function is continuous at the points where the definition changes. For example, if f(x) = 5 when x < 2 and f(x) = ax + b when x > 2, then continuity at x = 2 means the left-hand limit (5) should equal the right-hand limit (a*2 + b). So, 5 = 2a + b. By solving this equation, we relate a and b. If additional conditions like values of f at specific points or differentiability are given, those can be used to find precise values. For example, if f(x) = ax + 1 for x < 3 and f(x) = bx + 3 for x > 3, continuity at x = 3 requires that a*3 + 1 = b*3 + 3, leading to 3a + 1 = 3b + 3, which simplifies to 3a - 3b = 2. Combining this with any other conditions will give the exact values of a and b. Therefore, the method involves applying continuity conditions at boundary points and solving the resulting equations.

Question 33.

Find the intervals in which the function f(x) = x log/ x is strictly increasing or strictly decreasing.

[3 Marks]

Answer:

To find where the function f(x) = x log x is increasing or decreasing, we first determine its derivative. Using the product rule, the derivative f'(x) = 1 * log x + x * (1/x) = log x + 1. The natural logarithm function log x is defined for x > 0. Since f'(x) = log x + 1, the function is increasing where f'(x) > 0 and decreasing where f'(x) < 0.

Set f'(x) > 0: log x + 1 > 0 → log x > -1 → x > $e^{(-1)}$ = 1/e.

Set f'(x) < 0: log x + 1 < 0 → log x < -1 → x < 1/e.

Since the domain is x > 0, the function f(x) is strictly decreasing on (0, 1/e) and strictly increasing on (1/e, ∞).

Question 34.

Find the absolute maximum and absolute minimum values of the function f given by f(x) = x/2+ 2/x on the interval [1, 2].

[3 Marks]

Answer: To find the absolute maximum and minimum values of the function f(x) = x/2 + 2/x on the interval [1, 2], first find the derivative f'(x). Differentiating, f'(x) = 1/2 - $2/x^2$. Set f'(x) = 0 to find critical points: 1/2 - $2/x^2$ = 0, which gives x^2 = 4, hence x = 2 (only x = 2 lies in the interval [1, 2]). Now evaluate the function at the critical point and the endpoints: f(1) = 1/2 + 2/1 = 2.5, f(2) = 2/2 + 2/2 = 1 + 1 = 2. The function value at x=1 is 2.5, and at x=2 is 2. Since f(1) > f(2), the absolute maximum value is 2.5 at x = 1, and the absolute minimum value is 2 at x = 2.

Question 35.

Find: $\int x^2+1/(x^2+2)\ (x^2+4)\ dx$

[3 Marks]

Answer:

To solve the integral $\int (x^2 + 1) / ((x^2 + 2)(x^2 + 4))\ dx$, we start by expressing the integrand as a sum of partial fractions. Let

$(x^2 + 1) / ((x^2 + 2)(x^2 + 4)) = A / (x^2 + 2) + B / (x^2 + 4)$

Multiplying both sides by $(x^2 + 2)(x^2 + 4)$, we get:

$x^2 + 1 = A(x^2 + 4) + B(x^2 + 2) = (A + B)\ x^2 + (4A + 2B)$

Equating coefficients, we have:

For x^2: $1 = A + B$

Constant term: $1 = 4A + 2B$

Solving the two equations:

From the first: $B = 1 - A$

Substitute into the second: $1 = 4A + 2(1 - A) = 4A + 2 - 2A = 2A + 2$

So, $2A = -1 \Rightarrow A = -1/2$, and $B = 1 - (-1/2) = 3/2$

Thus, the integral becomes:

$\int (x^2 + 1)/((x^2 + 2)(x^2 + 4))\ dx = \int (-1/2) / (x^2 + 2)\ dx + \int (3/2) / (x^2 + 4)\ dx$

We know that $\int dx / (x^2 + a^2) = (1/a)\ \text{atan}(x/a) + C$.

Therefore,

Integral $= (-1/2) \int dx / (x^2 + 2) + (3/2) \int dx / (x^2 + 4)$

$= (-1/2) * (1/\sqrt{2})\ \text{atan}(x / \sqrt{2}) + (3/2) * (1/2)\ \text{atan}(x / 2) + C$

$= - (1 / (2 \sqrt{2}))\ \text{atan}(x / \sqrt{2}) + (3/4)\ \text{atan}(x / 2) + C$

Question 36.

Find $\int 2+\sin 2x\ /1 +\cos 2x\ e^x\ dx$.

[3 Marks]

Answer: To evaluate the integral ∫ [(2 + sin 2x) / (1 + cos 2x)] e^x dx, first simplify the expression inside the integral. Note that 1 + cos 2x can be written as 2 $\cos^2$ x, and sin 2x = 2 sin x cos x. Therefore, the integrand becomes [(2 + 2 sin x cos x) / (2 $\cos^2$ x)] e^x = [(1 + sin x / cos x) / cos x] e^x = (1 / cos x + tan x) e^x. The integral becomes ∫ (sec x + tan x) e^x dx. This integral can be solved using integration by parts or standard integration techniques, taking e^x as one function and (sec x + tan x) as another. Applying integration by parts or appropriate methods will lead to the solution, which involves expressions with e^x multiplied by trigonometric functions plus a constant of integration. Hence, the integral is solvable by simplifying the integrand using trigonometric identities and then applying integration methods involving e^x and trigonometric functions.

Question 37.

Evaluate

[3 Marks]

Answer:

To evaluate the given mathematical expressions, we need to apply suitable arithmetic operations step-by-step.

Part 1: Evaluate the determinant
| 2 4 |
|-1 2 |

The determinant of a 2x2 matrix |a b|
|c d| is calculated as (ad - bc).
So, determinant = (2 × 2) - (4 × -1) = 4 + 4 = 8.

Part 2: Evaluate the determinant
| 1 x + y y |
| 1 x x + y |

For this 2x3 matrix, since it is not a square matrix, the determinant is not defined. Possibly, the question may require clarification.

Part 3: Evaluate (1/3)^{-1} - (1/4)^{-1}, then find the inverse of the result.

(1/3)^{-1} = 3 and (1/4)^{-1} = 4, so 3 - 4 = -1. The inverse of -1 is -1.

Part 4: Evaluate (5/8)^{-7} × (8/5)^{-4}.

This is equal to (8/5)^{7} × (5/8)^{4} = (8/5)^{3} = 512 / 125.

Part 5: Evaluate factorial expressions 8! and 4! - 3!.

8! = 40320, 4! = 24 and 3! = 6, so 4! - 3! = 24 - 6 = 18.

Question 38.

Solve the following linear programming problem graphically:

Maximise $z = 4x + 3y$,

subject to the constraints

$x + y \leq 800$

$2x + y \leq 1000$

$x \leq 400$

$x, y \geq 0$.

[3 Marks]

Answer: To solve the linear programming problem graphically, first plot the constraints on the coordinate plane. Draw the lines for $x + y = 800$, $2x + y = 1000$, and $x = 400$. The feasible region is the area that satisfies all inequalities including $x \geq 0$ and $y \geq 0$. Identify the corner points of this feasible region by finding the intersection points of these lines, including points on the axes. Calculate the value of $z = 4x + 3y$ at each corner point. The maximum value of z among these points is the solution to the problem. This method uses the fact that the optimal value of a linear programming problem lies at a vertex of the feasible region.

Question 39.

The chances of P, Q and R getting selected as CEO of a company are in the ratio 4 : 1 : 2 respectively. The probabilities for the company to increase its profits from the previous year under the new CEO, P, Q or R are 0·3, 0·8 and 0·5 respectively. If the company increased the profits from the previous year, find the probability that it is due to the appointment of R as CEO.

[3 Marks]

Answer:

Let A be the event that the company increased profits, and let P, Q, and R represent the event that CEO P, Q, or R is selected respectively.

Given the ratio of chances for P, Q and R are 4 : 1 : 2, total parts = 4 + 1 + 2 = 7.

Probability of P selected, $P(P) = 4/7$; Probability of Q selected, $P(Q) = 1/7$; Probability of R selected, $P(R) = 2/7$.

Probabilities of increase in profits under each CEO are: $P(A|P) = 0.3$, $P(A|Q) = 0.8$, $P(A|R) = 0.5$.

Using total probability theorem:

$P(A) = P(P) \times P(A|P) + P(Q) \times P(A|Q) + P(R) \times P(A|R)$

$= (4/7 \times 0.3) + (1/7 \times 0.8) + (2/7 \times 0.5) = 0.171 + 0.114 + 0.143 = 0.428$

We need to find probability that increase in profits is due to R. That is, $P(R|A) = [P(R) \times P(A|R)] / P(A)$

$= (2/7 \times 0.5) / 0.428 = 0.143 / 0.428 \approx 0.334$

Hence, the probability that the increase in profits is due to the appointment of R as CEO is approximately 0.334 or 33.4%.

Section E

Question 40. A relation R on set $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$ is defined as $R = \{(x, y) : x + y$ is an integer divisible by $2\}$. Show that R is an equivalence relation. Also, write the equivalence class [2].

[5 Marks]

Answer: Given the set $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$ and the relation R defined by $R = \{(x, y) : x + y$ is divisible by $2\}$. To prove that R is an equivalence relation, we must show that it is reflexive, symmetric, and transitive. 1. Reflexive: For any x in A, $x + x = 2x$ which is divisible by 2, so (x, x) is in R. 2. Symmetric: If (x, y) is in R, then $x + y$ is even. Then $y + x$ is also even, so (y, x) is in R. 3. Transitive: If (x, y) and (y, z) are in R, then $x + y$ and $y + z$ are even. Adding these, $(x + y) + (y + z) = x + 2y + z$ is even. Since $2y$ is always even, this implies $x + z$ is even, so (x, z) is in R. Hence, R is an equivalence relation. The equivalence class $[2] = \{ y \text{ in } A \mid (2, y) \text{ in } R \}$ includes all y such that $2 + y$ is even. Since 2 is even, y must be even for $2 + y$ to be even. So, $[2] = \{-4, -2, 0, 2, 4\}$. This set groups all even numbers in A which are related to 2 under R.

Question 41. It is given that function $f(x) = x^4 - 6x^2 + ax + 9$ attains local maximum value at $x = 1$. Find the value of 'a', hence obtain all other points where the given function $f(x)$ attains local maximum or local minimum values.

[5 Marks]

Answer:

Given the function f(x) = x^4 – 6x^2 + a*x + 9, it is stated that it attains a local maximum at x = 1. To find the value of 'a', we first differentiate the function:

f'(x) = 4x^3 – 12x + a

For a local maximum at x = 1, the first derivative at x = 1 must be zero:

$f'(1) = 4(1)^3 - 12(1) + a = 4 - 12 + a = a - 8 = 0 \Longrightarrow a = 8$

Next, to confirm this is a maximum, we check the second derivative:

$f''(x) = 12x^2 - 12$

At x = 1, $f''(1) = 12(1)^2 - 12 = 12 - 12 = 0$, the second derivative test is inconclusive.

So, we analyze the critical points by solving f'(x) = 0 with a = 8:

$4x^3 - 12x + 8 = 0 \Longrightarrow x^3 - 3x + 2 = 0$

Factorizing, $(x - 1)^2(x + 2) = 0$. So the critical points are at x = 1 and x = −2.

Check second derivative at these points:

f''(1) = 0 (requires further test)

$f''(-2) = 12(4) - 12 = 48 - 12 = 36 > 0$, so local minimum at x = −2.

Use the first derivative test or check values around x = 1 to confirm the nature of x = 1. Also check for other critical points.

Thus, with a = 8, the function has a local maximum at x = 1 and a local minimum at x = −2.

Question 42. The perimeter of a rectangular metallic sheet is 300 cm. It is rolled along one of its sides to form a cylinder. Find the dimensions of the rectangular sheet so that volume of cylinder so formed is maximum.

[5 Marks]

Answer:

Let the length of the rectangular sheet be L cm and its breadth be B cm. The perimeter is given as 2(L + B) = 300 cm, so L + B = 150 cm.

When the sheet is rolled along one of its sides, the side chosen becomes the height and the other side forms the circumference of the cylinder's base. We consider two cases:

Case 1: Length is the height (h = L), breadth is the circumference (2πr = B)

Then, r = B / (2π) and volume $V = \pi r^2 h = \pi (B / 2\pi)^2 L = (B^2 L) / (4\pi)$

Using L = 150 - B, volume $V = (B^2 (150 - B)) / (4\pi) = (1 / 4\pi)(150B^2 - B^3)$

To maximize volume, differentiate V with respect to B and set dV/dB = 0:

$dV/dB = (1 / 4\pi)(300B - 3B^2) = 0$

$300B - 3B^2 = 0 \Rightarrow B(300 - 3B) = 0$

So $B = 0$ or $B = 100$. $B = 0$ is not possible, so $B = 100$ cm.

Therefore, $L = 150 - 100 = 50$ cm.

Height $h = L = 50$ cm, radius $r = B / (2\pi) = 100 / (2 \times 3.14) \approx 15.92$ cm.

Case 2: Breadth is the height ($h = B$), length is the circumference ($2\pi r = L$)

Similarly, $r = L / (2\pi)$, volume $V = \pi r^2 h = \pi (L / 2\pi)^2 B = (L^2 B) / (4\pi)$

Using $B = 150 - L$, $V = (1 / 4\pi) (L^2 (150 - L)) = (1 / 4\pi)(150L^2 - L^3)$

Differentiating w.r.t L:

$dV/dL = (1 / 4\pi)(300L - 3L^2) = 0 \Rightarrow L (300 - 3L) = 0 \Rightarrow L = 100$ cm or 0 cm (not possible)

$B = 150 - 100 = 50$ cm.

Height $h = B = 50$ cm, radius $r = L / (2\pi) = 100 / (2 \times 3.14) \approx 15.92$ cm.

Both cases give the same dimensions but rolled along different sides.

Final dimensions of the rectangular sheet: 100 cm by 50 cm.

These dimensions maximize the volume of the cylinder formed by rolling the sheet.

Question 43. Using integration, find the area of the region enclosed between the circle $x^2 + y^2 = 16$ and the lines $x = -2$ and $x = 2$.

[5 Marks]

Answer: To find the area of the region enclosed between the circle $x^2 + y^2 = 16$ and the vertical lines $x = -2$ and $x = 2$, we use integration. The circle has radius 4 since $16 = 4^2$. For each x in $[-2, 2]$, the y-values on the circle are given by $y = \pm\sqrt{(16 - x^2)}$. The area between the lines and the circle is twice the area under the upper semicircle from $x = -2$ to $x = 2$. This area can be expressed as $A = 2 \int$ from -2 to 2 of $\sqrt{(16 - x^2)}\, dx$. We calculate this integral using the standard formula for $\int\sqrt{(a^2 - x^2)}\, dx$, which is $(x/2)\sqrt{(a^2 - x^2)} + (a^2/2) \sin^{-1}(x/a) + C$. Applying this from -2 to 2, and doubling the result, gives the required area. Evaluating, we find $A = 2 * [(2/2)\sqrt{(16 - 4)} + (16/2) \sin^{-1}(2/4) - (-2/2)\sqrt{(16 - 4)} - (16/2) \sin^{-1}(-2/4)]$. Simplifying using $\sin^{-1}(0.5) = \pi/6$, and $\sqrt{12} = 2\sqrt{3}$, we get $A = 2 * [1 * 2\sqrt{3} + 8 * \pi/6 + 1 * 2\sqrt{3} + 8 * \pi/6] = 2 * [2\sqrt{3} + 8 * \pi/6 + 2\sqrt{3} + 8 * \pi/6] = 2 * [4\sqrt{3} + (16\pi)/6] = 2 * [4\sqrt{3} + (8\pi)/3] = 8\sqrt{3} + (16\pi)/3$. Thus, the area enclosed between the circle and the lines $x = -2$ and $x = 2$ is $8\sqrt{3} + (16\pi)/3$ square units.

Question 44.

Find the equation of the line passing through the point of intersection of the lines $x/1 = y-1/2=z-2/3$ and $x-1/0 = y/-3 = z-7/2$ and perpendicular to these given lines.

[5 Marks]

Answer:

First, find the point of intersection of the given two lines:

Line 1: $x/1 = (y-1)/2 = (z-2)/3 = t$

From Line 1, we have $x = t$, $y = 2t + 1$, $z = 3t + 2$.

Line 2: $(x-1)/0 = y/-3 = (z-7)/2$. Since $(x-1)/0$ is undefined unless $x = 1$, so $x = 1$.

From Line 2, $y/-3 = (z-7)/2 = s$

So, $y = -3s$, $z = 2s + 7$.

At point of intersection, x from Line 1 = x from Line 2 = 1, implies $t = 1$.

Put $t=1$ in Line 1: $y = 2(1) + 1 = 3$, $z = 3(1) + 2 = 5$. So point from Line 1: (1,3,5).

At intersection, y and z from Line 2 must be same as from Line 1:

$y = -3s = 3 \Rightarrow s = -1$.

$z = 2(-1) + 7 = 5$, matches z coordinate.

Thus, point of intersection is (1, 3, 5).

Now, find direction vectors of the two lines:

Line 1 direction vector: (1, 2, 3)

Line 2 direction vector: (0, -3, 2)

Direction vector of required line is perpendicular to both given lines, so it is the cross product of these two vectors.

Cross product = $(2*2 - 3*(-3), 3*0 - 1*2, 1*(-3) - 2*0) = (4 + 9, 0 - 2, -3 - 0) = (13, -2, -3)$

Equation of the required line passing through point (1, 3, 5) and having direction vector (13, -2, -3) is:

$(x - 1)/13 = (y - 3)/(-2) = (z - 5)/(-3)$

Question 45.

Two vertices of parallelogram ABCD are A(−1, 2, 1) and B(1, −2, 5). If the equation of the line passing through C and D is $x-4/1= y+7/-2 = z-8/2$, then find the distance between sides AB and CD. Hence, find the area of parallelogram ABCD.

[5 Marks]

Answer:

To find the distance between sides AB and CD and the area of parallelogram ABCD, we proceed as follows:

Step 1: Find the vector AB
Coordinates of A are $(-1, 2, 1)$ and B are $(1, -2, 5)$.
Vector AB = B - A = $(1 - (-1), -2 - 2, 5 - 1) = (2, -4, 4)$.

Step 2: Determine a point on line CD
The line passing through points C and D is given by:

$(x - 4)/1 = (y + 7)/-2 = (z - 8)/2 = t$ (parameter)

At $t = 0$, point C is $(4, -7, 8)$. At $t = 1$, point D is $(5, -9, 10)$. So, vector CD = D - C = $(1, -2, 2)$.

Step 3: Find the distance between lines AB and CD
Since AB and CD are sides of a parallelogram, they are parallel, which means vector AB is parallel to vector CD. Check their direction vectors:
Vector AB = $(2, -4, 4)$, vector CD = $(1, -2, 2)$. Vector AB = $2 \times$ vector CD, confirming parallelism.

The distance between the two sides is given by the distance between two parallel lines in space, which is given by the formula:
Distance = $|(\text{vector } AC) \cdot (n)| / |n|$, where n is a vector perpendicular to AB and CD, and vector AC is any vector connecting a point on AB to a point on CD.

We take vector AC = C - A = $(4 - (-1), -7 - 2, 8 - 1) = (5, -9, 7)$.

To find the vector n perpendicular to AB and CD, we can take a vector that is perpendicular to AB and CD, such as the cross product of AB and any vector perpendicular to AB; however, since AB and CD are parallel, the normal vector perpendicular to both sides is any vector perpendicular to AB.

But since the distance is the projection of AC on a vector perpendicular to AB, the distance can be found using the formula:

Distance = $|(AC \times AB)| / |AB|$.

Calculate $AC \times AB$:
AC = $(5, -9, 7)$
AB = $(2, -4, 4)$

$AC \times AB = |i\ j\ k|$
$|5\ {-9}\ 7|$
$|2\ {-4}\ 4|$

$= i((-9)(4) - 7(-4)) - j(5(4) - 7(2)) + k(5(-4) - (-9)(2))$
$= i(-36 + 28) - j(20 - 14) + k(-20 + 18)$

= i(-8) - j(6) + k(-2) = (-8, -6, -2).

Now, |AC × AB| = sqrt((-8)^2 + (-6)^2 + (-2)^2) = sqrt(64 + 36 + 4) = sqrt(104) ≈ 10.20.

Also, |AB| = sqrt(2^2 + (-4)^2 + 4^2) = sqrt(4 + 16 + 16) = sqrt(36) = 6.

Therefore, distance between AB and CD = (|AC × AB|) / |AB| = 10.20 / 6 = 1.7 units.

Step 4: Find the area of the parallelogram ABCD
The area of a parallelogram defined by vectors AB and AD is given by the magnitude of their cross product. Here, since AB and CD are parallel sides, we can use AB as base and the distance calculated as height:
Area = base × height = |AB| × distance = 6 × 1.7 = 10.2 square units.

Final Answer:
Distance between sides AB and CD = 1.7 units.
Area of parallelogram ABCD = 10.2 square units.

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