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Patterns in Consecutive Square Numbers

This section explores a pattern observed in three consecutive square numbers. Consider three consecutive integers $n - 1$, n, and $n + 1$ and their squares $(n - 1)^2$, n^2, and $(n + 1)^2$.

Formula Derivation

Sum the smallest and largest squares and subtract twice the middle square:

$(n - 1)^2 + (n + 1)^2 - 2n^2$

Expanding each term:

$(n^2 - 2n + 1) + (n^2 + 2n + 1) - 2n^2 = n^2 - 2n + 1 + n^2 + 2n + 1 - 2n^2$

Simplify:

$= (n^2 + n^2 - 2n^2) + (-2n + 2n) + (1 + 1) = 0 + 0 + 2 = 2$

Thus, the result is always 2 for any integer n.

Worked Illustration

- For $n = 2$, squares are 1, 4, 9. Calculation: $1 + 9 - 2 \times 4 = 10 - 8 = 2$.
- For $n = 4$, squares are 9, 16, 25. Calculation: $9 + 25 - 2 \times 16 = 34 - 32 = 2$.
- For $n = 6$, squares are 25, 36, 49. Calculation: $25 + 49 - 2 \times 36 = 74 - 72 = 2$.

Practice Set

- Level 1: Verify the pattern for $n = 3, 5, 7$.
- Level 2: Prove the pattern for four consecutive squares $(n - 1)^2$, n^2, $(n + 1)^2$, $(n + 2)^2$.
- Level 3: Explore if a similar pattern exists for cubes of three consecutive numbers.

Answer Key

- Level 1: The result is always 2.
- Level 2: Requires algebraic expansion and simplification.
- Level 3: Investigation and proof required.

Quick Reference

$(n - 1)^2 + (n + 1)^2 - 2n^2 = 2$

Glossary

- **Consecutive numbers:** Numbers that follow each other in order.
- **Square of a number:** The product of the number with itself.

Visualising Algebraic Identities

This section uses geometric models to understand algebraic identities, particularly squares and rectangles representing terms.

Concept Explanation

Consider two line segments of lengths a and b. Construct a square of side a + b and partition it into smaller squares and rectangles.

Formula Derivation

The area of the large square is $(a + b)^2$. Partitioning it gives:

- Square of side a with area a^2.
- Square of side b with area b^2.
- Two rectangles each with area ab.

Summing these areas:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Worked Illustration

- Example: Expand $(5x + 2y)^2 = 25x^2 + 20xy + 4y^2$.
- Calculate 43^2 using the identity: $(40 + 3)^2 = 1600 + 240 + 9 = 1849$.

Practice Set

- Level 1: Expand $(3a + 4b)^2$.
- Level 2: Calculate 57^2 using the identity.
- Level 3: Factor expressions like $x^2 + 6x + 9$ using the identity.

Answer Key

- Level 1: $9a^2 + 24ab + 16b^2$.
- Level 2: $57^2 = (50 + 7)^2 = 2500 + 700 + 49 = 3249$.
- Level 3: $x^2 + 6x + 9 = (x + 3)^2$.

Quick Reference

$(a + b)^2 = a^2 + 2ab + b^2$

Glossary

- **Algebraic identity:** An equation true for all values of variables.
- **Binomial:** An algebraic expression with two terms.

Expansion and Factorisation of Algebraic Expressions

This section covers the use of algebraic identities to expand and factor expressions.

Concept Explanation

Using identities like $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$ to expand expressions and factor quadratic expressions.

Formula Derivation

Example of factorisation:

$x^2 + 4x + 4 = (x + 2)^2$

By comparing with $a^2 + 2ab + b^2$, where $a = x$ and $b = 2$.

Worked Illustrations

- Factor $36x^2 + 12x + 1 = (6x + 1)^2$.
- Factor $50p^2 + 60pq + 18q^2 = 2(5p + 3q)^2$.

Practice Set

- Level 1: Factor $x^2 + 7x + 12$.
- Level 2: Factor $x^2 - 5x + 6$.
- Level 3: Factor $4x^2 + 12x + 9$.

Answer Key

- Level 1: $(x + 3)(x + 4)$.
- Level 2: $(x - 2)(x - 3)$.
- Level 3: $(2x + 3)^2$.

Quick Reference

$x^2 + (a + b)x + ab = (x + a)(x + b)$

Glossary

- **Factorisation:** Expressing an expression as a product of simpler expressions.
- **Quadratic expression:** Polynomial of degree 2.

Cube of a Binomial

This section introduces the identity for the cube of a binomial and its geometric visualization.

Formula Derivation

Using distributive property:

$(a + b)^3 = (a + b)(a + b)^2 = (a + b)(a^2 + 2ab + b^2) = a^3 + 3a^2b + 3ab^2 + b^3$

Similarly,

$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Worked Illustration

- Find the side of a cube with volume $p^3 + 6p^2q + 12pq^2 + 8q^3$ by comparing with $(a + b)^3$.
- Express $8n^3 - 60n^2m + 150nm^2 - 125m^3$ as $(a - b)^3$.

Practice Set

- Level 1: Expand $(x + 2)^3$.
- Level 2: Factor $27a^3 + 64b^3$.
- Level 3: Factor $125x^3 - 216y^3$.

Answer Key

- Level 1: $x^3 + 6x^2 + 12x + 8$.
- Level 2: $(3a + 4b)^3$.
- Level 3: $(5x - 6y)^3$.

Quick Reference

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Glossary

- **Binomial cube:** Cube of a binomial expression.
- **Cuboid:** A three-dimensional rectangular box.

Algebraic Identities Involving Three Variables

This section explores identities involving three variables such as $(a + b + c)^2$ and the factorisation of sums and differences of cubes.

Formula Derivation

For three variables a, b, c,

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

Also,

$(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = x^3 + y^3 + z^3 - 3xyz$

Worked Illustration

- Calculate the sum of cubes of three numbers given their sum, product, and sum of squares.

Practice Set

- Level 1: Expand $(x + y + z)^2$.
- Level 2: Verify the identity for specific values of x, y, z.
- Level 3: Use the identity to solve problems involving sums and products of three variables.

Answer Key

- Level 1: $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$.
- Level 2: Verification by substitution.
- Level 3: Application of the identity to find unknown sums.

Quick Reference

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = x^3 + y^3 + z^3 - 3xyz$

Glossary

- **Sum of cubes:** The sum of the cubes of variables.

- **Symmetric expression:** Expression unchanged by permutation of variables.

Simplification of Rational Algebraic Expressions

This section explains how to simplify rational expressions by factorisation and cancelling common factors.

Concept Explanation

Given a rational expression P(x)/Q(x), simplify by factoring numerator and denominator and cancelling common factors where Q(x) not equal to 0.

Worked Illustration

- Simplify $(x^2 - 7x + 12)/(5x^2 + 5x - 100)$:
  - Factor numerator: $x^2 - 7x + 12 = (x - 3)(x - 4)$.
  - Factor denominator: $5x^2 + 5x - 100 = 5(x^2 + x - 20) = 5(x - 4)(x + 5)$.
  - Cancel common factor x - 4 (assuming x not equal to 4):
  - Result: $(x - 3)/5(x + 5)$.

Practice Set

- Level 1: Simplify $(36s^2 - 12st + t^2)/(t^2 + 2ts - 48s^2)$.
- Level 2: Simplify $(x^2 + 5x + 6)/(x^2 - 4)$.
- Level 3: Simplify $(4x^2 - 9)/(2x^2 + 7x + 3)$.

Answer Key

- Level 1: $((6s - t)^2)/((t - 6s)(t + 8s))$.
- Level 2: $((x + 2)(x + 3))/((x - 2)(x + 2)) = (x + 3)/(x - 2)$.

- Level 3: Factor and simplify accordingly.

Quick Reference

Simplify rational expressions by factoring numerator and denominator and cancelling common factors.

Glossary

- **Rational expression:** Ratio of two polynomials.
- **Factor:** Expression multiplied to get another expression.