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Natural Numbers

Natural numbers are the basic counting numbers used since ancient times for practical counting needs. They are denoted by N = {1, 2, 3, 4, ...}. Early humans used one-to-one correspondence to count objects, such as matching pebbles to cattle to ensure none were lost.

Formula Derivation

Natural numbers are the foundation of the number system and do not require derivation but serve as the building blocks for other number sets.

Worked Illustrations

Counting 5 apples: 1, 2, 3, 4, 5.

Solved Examples

Example: Count the number of cows if there are 7 cows in the herd.

Solution: The number of cows is 7, a natural number.

Practice Set

- Level 1: List the first 10 natural numbers.
- Level 2: Find the sum of the first 20 natural numbers.
- Level 3: Prove that the sum of the first n natural numbers is $n(n+1)/2$.

Answer Key

- Level 1: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
- Level 2: $(20 \times 21)/2 = 210$
- Level 3: Use induction or formula derivation for sum of natural numbers.

Quick Reference

Natural numbers are positive integers starting from 1.

Glossary

- **Natural Numbers:** Counting numbers starting from 1.
- **One-to-One Correspondence:** Matching each element of one set to exactly one element of another set.

Zero and Negative Numbers

The concept of zero (Sunya) was formalized in India and introduced into mathematics by Brahmagupta in 628 CE. Zero represents the absence of quantity and allows arithmetic operations involving nothingness.

Formula Derivation

Brahmagupta defined zero as the result of subtracting a number from itself: $a - a = 0$.

Worked Illustrations

Adding zero: $a + 0 = a$

Multiplying by zero: $a \times 0 = 0$

Solved Examples

Example: Calculate $7 + 0$ and 5×0.

Solution: $7 + 0 = 7$, $5 \times 0 = 0$.

Practice Set

- Level 1: Compute $10 - 10$.
- Level 2: Evaluate 0×15.
- Level 3: Prove that $a + 0 = a$ for any integer a.

Answer Key

- Level 1: 0
- Level 2: 0
- Level 3: Use the definition of zero and properties of addition.

Quick Reference

Zero is the additive identity: adding zero to any number leaves it unchanged.

Glossary

- **Zero (Sunya):** The number representing nothingness.
- **Negative Numbers:** Numbers less than zero, introduced as debts by Brahmagupta.

Integers

Integers (Z) include all natural numbers, zero, and negative numbers: {..., -3, -2, -1, 0, 1, 2, 3, ...}. Brahmagupta introduced negative numbers as debts (Rina) and positive numbers as fortunes (Dhana).

Formula Derivation

Brahmagupta's rules for integers include:

- $a + 0 = a$
- $a - 0 = a$
- $a \times 0 = 0$
- Sum of two positive integers is positive.
- Sum of two negative integers is negative.
- Product of two negative integers is positive.

Solved Examples

Example: Calculate (-3) + (-4) and (-3) × (-4).

Solution: (-3) + (-4) = -7, (-3) × (-4) = 12.

Practice Set

- Level 1: Add 5 + (-2).
- Level 2: Multiply (-6) × 3.
- Level 3: Prove that the product of two negative integers is positive.

Answer Key

- Level 1: 3
- Level 2: -18
- Level 3: Use the concept of debts and removal of debts.

Quick Reference

Integers extend natural numbers to include zero and negatives.

Glossary

- **Integers (Z):** Whole numbers including negatives and zero.
- **Debts (Rina):** Negative numbers representing owed quantities.
- **Fortunes (Dhana):** Positive numbers representing assets.

Rational Numbers

Rational numbers (Q) are numbers that can be expressed as p/q, where p, q in Z and q not equal to 0. They include integers and fractions, both positive and negative.

Formula Derivation

Operations on rational numbers:

- Equality: a/b = c/d if and only if ad = bc
- Addition/Subtraction: a/b plus or minus c/b = (a plus or minus c)/b
- Multiplication: a/b × c/d = (ac)/(bd)
- Division: (a/b) ÷ (c/d) = (a/b) × (d/c) = (ad)/(bc)

Solved Examples

Example: Add 2/3 + 1/3.

Solution: 2/3 + 1/3 = 3/3 = 1.

Practice Set

- Level 1: Simplify 4/6.
- Level 2: Multiply 3/4 × 2/5.
- Level 3: Prove that rational numbers are dense on the number line.

Answer Key

- Level 1: 2/3

- Level 2: 6/20 = 3/10
- Level 3: Use the average of two rational numbers to find another rational number between them.

Quick Reference

Rational numbers are fractions with integer numerator and non-zero integer denominator.

Glossary

- **Rational Numbers (Q):** Numbers expressible as p/q with integers p, q and q not equal to 0.
- **Co-prime:** Two integers with no common factors other than 1.

Representation of Rational Numbers on the Number Line

To represent a rational number p/q on the number line, divide the unit interval between two integers into q equal parts and move p parts from zero to the right if positive, or to the left if negative.

Worked Illustrations

Represent 3/4 by dividing the interval between 0 and 1 into 4 equal parts and moving 3 parts to the right.

Solved Examples

Example: Locate 9/4 on the number line.

Solution: 9/4 = 2 1/4, so divide the interval between 2 and 3 into 4 parts and move 1 part to the right of 2.

Practice Set

- Level 1: Represent 1/2 on the number line.
- Level 2: Represent -3/4 on the number line.
- Level 3: Represent 8/5 and -7/4 on the number line.

Answer Key

- Level 1: Halfway between 0 and 1.
- Level 2: Three quarters to the left of 0.
- Level 3: 8/5 = 1 3/5 right of 0; -7/4 = -1 3/4 left of 0.

Quick Reference

Rational numbers can be precisely located on the number line by dividing intervals.

Glossary

- **Number Line:** A straight line with points representing numbers in order.
- **Unit Interval:** The distance between two consecutive integers on the number line.

Absolute Value of Rational Numbers

The absolute value |x| of a rational number x is its distance from zero on the number line, always non-negative.

Worked Illustrations

|5/3| = 5/3, |-5/3| = 5/3, |0| = 0.

Solved Examples

Example: Find the distance between -4 and 3 on the number line.

Solution: Distance = |-4 - 3| = |-7| = 7.

Practice Set

- Level 1: Find |-2|.
- Level 2: Calculate the distance between 5 and -1.
- Level 3: Prove that |a - b| represents the distance between two rational numbers a and b.

Answer Key

- Level 1: 2
- Level 2: 6
- Level 3: Use properties of absolute value and number line distance.

Quick Reference

Absolute value measures distance from zero, ignoring sign.

Glossary

- **Absolute Value:** Non-negative value representing distance from zero.

Density of Rational Numbers

Rational numbers are dense on the number line, meaning between any two rational numbers, there exists another rational number.

Worked Illustrations

Between 1 and 2, the number 3/2 lies; between 1 and 3/2, the number 5/4 lies.

Solved Examples

Example: Find a rational number between 1/3 and 1/2.

Solution: Average = (1/3 + 1/2)/2 = 5/12.

Practice Set

- Level 1: Find a rational number between 3 and 4.
- Level 2: Find a rational number between 2/5 and 3/5.
- Level 3: Prove that the average of two rational numbers lies between them.

Answer Key

- Level 1: 3.5 or 7/2
- Level 2: 5/10 = 1/2
- Level 3: Use inequalities and properties of averages.

Quick Reference

Between any two rational numbers, there is always another rational number.

Glossary

- **Density:** Property of rational numbers having infinitely many numbers between any two.

Irrational Numbers

Irrational numbers cannot be expressed as a ratio of two integers. Their decimal expansions are non-terminating and non-repeating. Examples include square root of 2, pi, and square root of 10.

Proof of Irrationality of square root of 2

Step 1: Assume square root of 2 = p/q in simplest form.

Step 2: Square both sides: 2 = p squared / q squared.

Step 3: Multiply both sides by q squared: 2q squared = p squared.

Step 4: Since p squared is even, p is even; let p = 2k.

Step 5: Substitute: 2q squared = 4k squared implies q squared = 2k squared.

Step 6: Thus, q is even.

Step 7: Both p and q are even, contradicting simplest form assumption.

Therefore, square root of 2 is irrational.

Practice Set

- Level 1: State whether square root of 3 is rational or irrational.
- Level 2: Prove irrationality of square root of 3 using contradiction.
- Level 3: Explore irrationality of square root of 5, square root of 7, and square root of 10.

Answer Key

- Level 1: Irrational
- Level 2: Similar proof by contradiction as for square root of 2.
- Level 3: Use prime factorization and contradiction.

Quick Reference

Irrational numbers cannot be expressed as fractions and have non-repeating decimals.

Glossary

- **Irrational Numbers:** Numbers not expressible as p/q.
- **Proof by Contradiction:** A method assuming the opposite to find a contradiction.

Construction of Irrational Lengths

To construct a line segment of length square root of 2 on the number line:

1. Mark points O = 0 and A = 1 on the number line.
2. Draw a perpendicular at A and mark B such that AB = 1.
3. Join O to B. By Pythagoras theorem, OB = square root of 2.
4. Using a compass, draw an arc with center O and radius OB intersecting the number line at P.
5. Point P represents square root of 2 on the number line.

Practice Set

- Level 1: Construct square root of 2 on a number line.
- Level 2: Construct square root of 3 and square root of 5 using the same method.
- Level 3: Generalize construction for square root of n, where n is a positive integer.

Answer Key

- Level 1: Follow the steps above.
- Level 2: Use perpendiculars and Pythagoras theorem accordingly.
- Level 3: Use successive perpendicular constructions for square root of n.

Quick Reference

Use right triangles and Pythagoras theorem to construct irrational lengths.

Glossary

- **Pythagoras Theorem:** a squared + b squared = c squared for right triangles.
- **Compass:** Tool to draw arcs and circles.

Irrational Numbers, Pi, and Infinite Series

Pi (pi) is an irrational number representing the ratio of a circle's circumference to its diameter. Aryabhata approximated pi approximately equal to 3927/1250 = 3.1416.

Madhava discovered the infinite series for pi:

pi = 4 × (1 - 1/3 + 1/5 - 1/7 + ...)

This infinite sum approaches pi as more terms are added.

Practice Set

- Level 1: Calculate the first 3 terms of the series.
- Level 2: Approximate pi using 5 terms.
- Level 3: Explore convergence of infinite series.

Answer Key

- Level 1: 4 × (1 - 1/3 + 1/5) = 4 × 13/15 = 3.4667
- Level 2: 4 × (1 - 1/3 + 1/5 - 1/7 + 1/9) approximately 3.3397
- Level 3: Series converges to pi as terms increase.

Quick Reference

Infinite series can represent irrational numbers like pi.

Glossary

- **Infinite Series:** Sum of infinitely many terms.
- **Convergence:** Approaching a finite value as terms increase.

Decimal Expansions of Rational and Irrational Numbers

Rational numbers have decimal expansions that either terminate or repeat. Irrational numbers have non-terminating, non-repeating decimals.

Terminating Decimals

Example: 3/8 = 0.375

Repeating Decimals

Example: 5/11 = 0.454545... = 0.45 repeating

Conversion of Repeating Decimals to Fractions

Example: Convert 0.6 repeating to fraction.

Let x = 0.6 repeating.

Multiply by 10: 10x = 6.6 repeating.

Subtract: 10x - x = 6.6 repeating - 0.6 repeating = 6.

So, 9x = 6 implies x = 6/9 = 2/3.

Practice Set

- Level 1: Convert 0.3 repeating to fraction.
- Level 2: Convert 0.45 repeating to fraction.
- Level 3: Convert 2.357 repeating to fraction.

Answer Key

- Level 1: 1/3
- Level 2: 5/11
- Level 3: 2122/900 = 1061/450

Quick Reference

Repeating decimals can be converted to fractions using algebraic manipulation.

Glossary

- **Repeating Decimal:** Decimal with a repeating block of digits.
- **Terminating Decimal:** Decimal that ends after finite digits.

Cyclic Numbers

Cyclic numbers are repeating decimal blocks with special properties. For example, 1/7 = 0.142857 repeating where multiplying 142857 by 1 to 6 cycles the digits.

Practice Set

- Level 1: Write the repeating block of 1/7.
- Level 2: Multiply 142857 by 3 and observe the pattern.
- Level 3: Explore cyclic properties of other fractions.

Answer Key

- Level 1: 142857
- Level 2: 428571 (digits cycle)
- Level 3: Research other cyclic numbers.

Quick Reference

Cyclic numbers exhibit digit rotations upon multiplication.

Glossary

- **Cyclic Number:** Number whose multiples are cyclic permutations of its digits.

Real Numbers

Real numbers (R) include all rational and irrational numbers, forming a continuous number line.

Summary

- Natural Numbers subset Integers subset Rational Numbers subset Real Numbers.
- Irrational numbers fill the gaps between rational numbers.
- Decimal expansions distinguish rational (terminating/repeating) and irrational (non-repeating) numbers.

Practice Set

- Level 1: Identify if a number is rational or irrational.
- Level 2: Place pi approximately on the number line.
- Level 3: Explain the difference between rational and irrational numbers.

Answer Key

- Level 1: Based on decimal expansion or fraction form.
- Level 2: pi approximately 3.1416 between 3 and 4.
- Level 3: Rational numbers can be expressed as fractions; irrational cannot.

Quick Reference

Real numbers form the complete continuous number line.

Glossary

- **Real Numbers (R):** All rational and irrational numbers.

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