Theorems for free!

Lars Hupel
MuniHac
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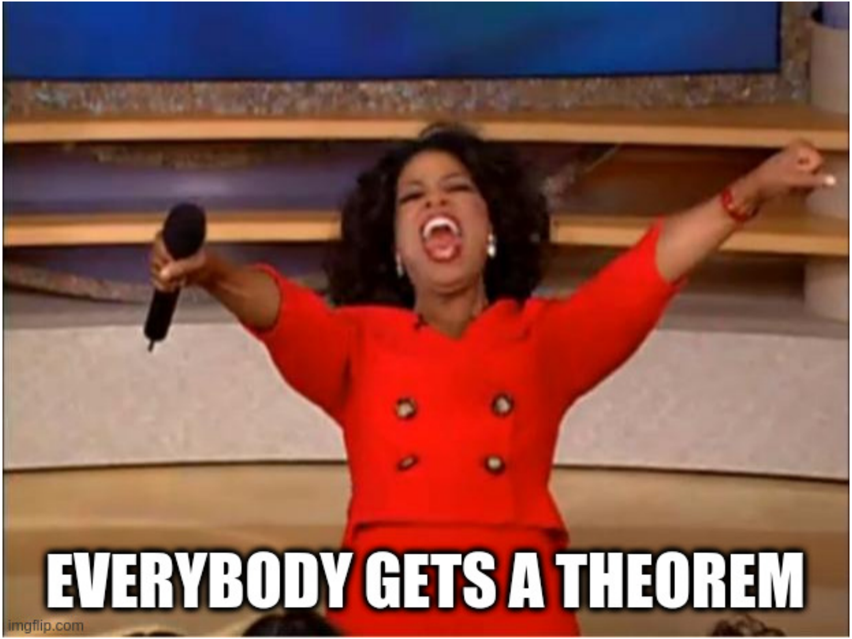
EVERYBODY GETS A THEOREM
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Types in Haskell

Type basics

- type variables are lower case
- all types are erased

(ignoring classes for now)

What Haskell sees:

```
id :: a -> a
```

What Haskell sees:

```
id :: a -> a
```

What the runtime sees:

```
id :: Word -> Word
```

Folklore says:

The more type variables, the merrier!

```haskell
data Lens s a = Lens
  { getter :: s -> a
  , setter :: a -> s -> s }
```

```haskell
type Lens s t a b =
  Functor f =>
    (a -> f b) ->
    s -> f t
```

More type variables!

… but why?

We can reason about types!

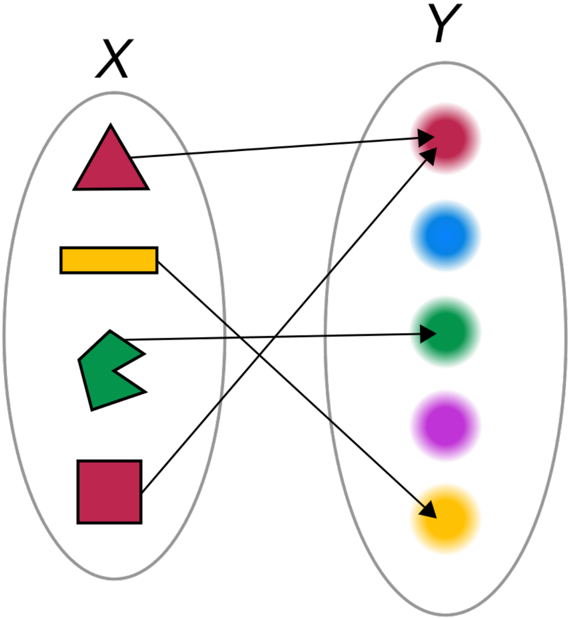
… but how?

Sets in mathematics

In set theory, everything[1] is a set.

For example: $\mathbb{N} = \{0, 1, 2, \ldots\}$

[1] almost


X
Y

Functions are sets

$$f = \{(\blacktriangle, \bullet), (\blacksquare, \bullet), \ldots\}$$

Types are sets

$$\begin{aligned}
[\![\texttt{Bool}]\!] &= \{\texttt{True}, \texttt{False}\} \\
[\![\texttt{Integer}]\!] &= \{\ldots, -2, -1, 0, 1, 2, \ldots\} \\
[\![(a, b)]\!] &= [\![a]\!] \times [\![b]\!] \\
[\![a \to b]\!] &= \text{the set of all functions from } [\![a]\!] \text{ to } [\![b]\!]
\end{aligned}$$

Key insight:

There are many different interpretations of types.

Side note

Wadler's paper uses A^* instead of `[a]`. Any idea why?

Relations

Relation R between A and B: $R \subseteq A \times B$


always have been
wait, types are relations too?
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Types are relations

We can assign every type t a relation rel_t.

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We can assign every type t a relation rel_t.

This relation will relate values of $[\![t]\!]$: $\mathrm{rel}_t \subseteq [\![t]\!] \times [\![t]\!]$

Ground types

… are identity relations

$$\begin{aligned} \mathsf{rel}_{\texttt{Bool}} &= \{(\texttt{True}, \texttt{True}), (\texttt{False}, \texttt{False})\} \\ \mathsf{rel}_{\texttt{Integer}} &= \{(n, n) \mid n \in \mathbb{Z}\} \end{aligned}$$

Lists

We have a relation for a.
We want to check if $\mathrm{xs}, ys : [a]$ are related.

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We have a relation for a.
We want to check if $\text{xs}, \text{ys} : [a]$ are related.

$\longrightarrow$ xs and ys need to be the same length and pairwise related

Lists: example

Let $\text{rel}_a\ x\ y = (y = 2 \cdot x)$

$[]$	$[]$	✓
$[1, 2]$	$[2, 4]$	✓
$[1, 2]$	$[2, 4, 6]$	✗
$[1, 2]$	$[0, 1]$	✗

Functions

When are two functions related?

When they send related inputs to related outputs.

Functions

$f : a \to b$ and $g : a \to b$ are related if:

$$\forall x, y \in [\![a]\!].\ (x, y) \in \mathrm{rel}_a \implies (f\ x, g\ y) \in \mathrm{rel}_b$$

Parametricity

The parametricity theorem

If t is a closed term of type T, then $(t, t) \in \mathrm{rel}_T$.

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In other words: every term is related to itself

Let's say we have a function on maps.

```
frobnicate :: [a] -> [a]
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We can prove:

$$\texttt{frobnicate}\ (\texttt{map}\ g\ xs) = \texttt{map}\ g\ (\texttt{frobnicate}\ xs)$$

Now what?


BEFORE

AFTER

Reasoning about types

Motto: Functions with type variables ...

- don't know anything
- can't do much

In practise

The second `Functor` law is redundant.

It is sufficient to prove that `fmap id = id`.

Free Theorems!

Please enter a (polymorphic) type, e.g. "(a -> Bool) -> [a] -> [a]":

(a -> Bool) -> [a] -> [a]

Please choose a sublanguage of Haskell:

no bottoms (hence no general recursion and no selective strictness)

inequational theorems (only relevant in a language with bottoms)

hide type instantiations in the theorem presentation

The Free Theorem

```
forall t1,t2 in TYPES, R in REL(t1,t2).
 forall p :: t1 -> Bool.
  forall q :: t2 -> Bool.
   (forall (x, y) in R. p x = q y)
   ==> (forall (z, v) in lift{[]}(R). (f p z, f q v) in lift{[]}(R))
```

The Free Theorem

with all permissable relation variables reduced to functions

```
forall t1,t2 in TYPES, g :: t1 -> t2.
 forall p :: t1 -> Bool.
  forall q :: t2 -> Bool.
   (forall x :: t1. p x = q (g x))
   ==> (forall y :: [t1]. map g (f p y) = f q (map g y))
```

Another free theorem

A function with type `(a -> b) -> [a] -> [b]` is either

1. `map`, or
2. `map` with rearrangements

Restrictions

$\bot$ destroys everything[2]

---

[2] not everything

Extensions

We have ignored classes (so far) because they complicate things.

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Classes can be modelled as dictionaries with (potentially) rank-2 types

Q & A

INNOQ
www.innoq.com

Lars Hupel

lars.hupel@innoq.com

@larsr_h

innoQ Deutschland GmbH

Krischerstr. 100
40789 Monheim a. Rh.
Germany
+49 2173 3366-0

Ohlauer Str. 43
10999 Berlin
Germany

Ludwigstr. 180 E
63067 Offenbach
Germany

Kreuzstr. 16
80331 München
Germany

c/o WeWork
Hermannstrasse 13
20095 Hamburg
Germany

innoQ Schweiz GmbH

Gewerbestr. 11
CH-6330 Cham
Switzerland
+41 41 743 01 11

Albulastr. 55
8048 Zürich
Switzerland

LARS HUPEL

Consultant
innoQ Deutschland GmbH

Lars enjoys programming in a variety of languages, including Scala, Haskell, and Rust. He is known as a frequent conference speaker and one of the founders of the Typelevel initiative which is dedicated to providing principled, type-driven Scala libraries.

Credits

- John C. Reynolds: `https://commons.wikimedia.org/w/index.php?title=File:Reynolds_John_small.jpg&oldid=452226049`, Andrej Bauer, CC-BY-SA 2.5
- Philip Wadler: `https://commons.wikimedia.org/w/index.php?title=File:Wadler2.JPG&oldid=262214892`, Clq, CC-BY 3.0
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- Free Theorems: `https://free-theorems.nomeata.de/`, Joachim Breitner et al.