Techniques checklist

0. Basics

   - ☐ Briefly understand what a ***function*** is
   - ☐ Determine the ***domain*** of a function
   - ☐ Determine the ***range*** of a function

1. **Inverse functions**

   - ☐ Determine if an inverse function f^{-1} ***exists***
   - ☐ Formulas for ***domain and range*** $D_{f^{-1}}$ and $R_{f^{-1}}$
   - ☐ Find the ***rule*** for $f^{-1}(x)$
   - ☐ ***Relationship*** between the graphs of $y = f(x)$ and $y = f^{-1}(x)$

2. **Composite functions**

   - ☐ Determine if a composite function fg ***exists***
   - ☐ Formula for ***domain*** D_{fg}
   - ☐ Finding the ***range*** R_{fg}
   - ☐ Find the ***rule*** for $fg(x)$

3. Miscellaneous

   - ☐ $f^2(x) = ff(x)$
   - ☐ $ff^{-1}(x)$, $f^{-1}f(x)$
   - ☐ "Self-inverse" functions
   - ☐ Piecewise functions
   - ☐ Repeating/periodic functions

1 Inverse functions

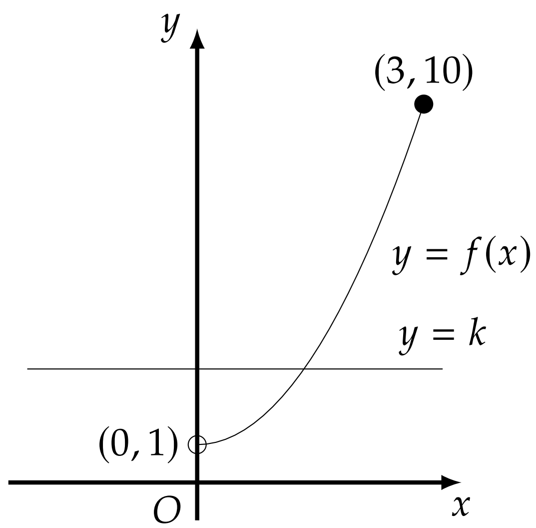
1.1 Existence of inverse functions

1.1.1 Inverse function exists

Example 1.1A: Horizontal line test (inverse exists)

The function f is defined by

$$f : x \mapsto x^2 + 1, \qquad x \in \mathbb{R}, -2 < x \leq 3.$$



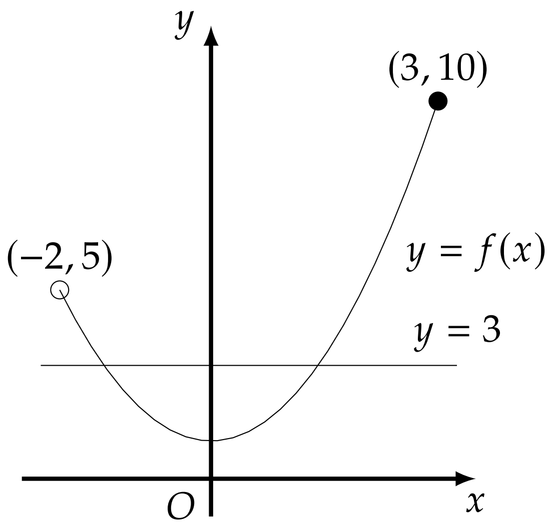
All horizontal lines $y = k, k \in \mathbb{R}$ cut the curve $y = f(x)$ ***at most once***. Hence f ***is one-one*** and f^{-1} exists. ■

1.1.2 Inverse function does not exist

Example 1.1B: Horizontal line test (inverse does not exist)

The function f is defined by

$$f : x \mapsto x^2 + 1, \qquad x \in \mathbb{R}, -2 < x \leq 3.$$



The horizontal line $y = 3$ cuts the curve $y = f(x)$ ***more than once***. Hence f ***is not one-one*** and does not have an inverse. ■

1.1.3 Restriction of domain to get an inverse function

Example 1.1C: Restriction of domain

The function f is defined by

$$f : x \mapsto x^2 + 1, \qquad x \in \mathbb{R}, -2 < x \leq b.$$

Find the largest value of b such that f^{-1} exists.

Answer: Largest $b = 0$. ■

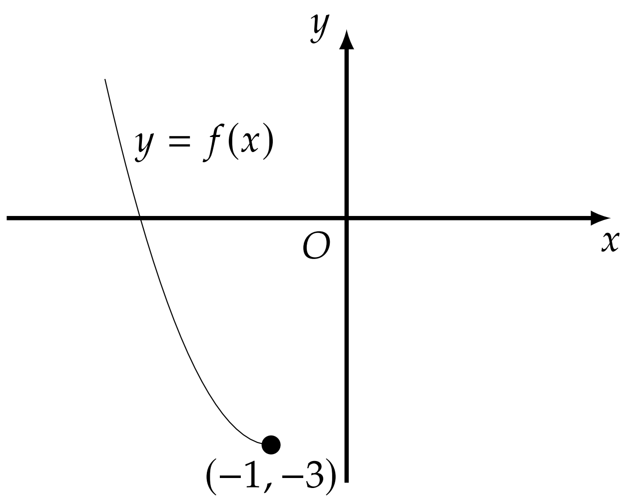
1.2 Formulas for $D_{f^{-1}}$ and $R_{f^{-1}}$

- $D_{f^{-1}} = R_f$
- $R_{f^{-1}} = D_f$

Example 1.2: $D_{f^{-1}}$ **and** $R_{f^{-1}}$

The function f is defined by

$$f : x \mapsto x^2 + 2x - 2, \qquad x \in \mathbb{R}, x \leq -1.$$



$D_{f^{-1}} = R_f = [-3, \infty)$. ■
$R_{f^{-1}} = D_f = (-\infty, -1]$. ■

1.3 Rule for f^{-1}

Example 1.3: Finding $f^{-1}(x)$

The function f is defined by

$$f : x \mapsto x^2 + 2x - 2, \qquad x \in \mathbb{R}, x \leq -1.$$

Let $y = x^2 + 2x - 2$
$y = (x+1)^2 - 3$
$(x+1)^2 = y + 3$
$x + 1 = \pm\sqrt{y+3}$

Since $x \leq -1$ (domain of f),
$x = -1 - \sqrt{y+3}$
Hence $f^{-1}(x) = -1 - \sqrt{x+3}$. ■

Definition of f^{-1}:

$$f^{-1} : x \mapsto -1 - \sqrt{x+3}, \qquad x \in \mathbb{R}, x \geq -3.$$

■

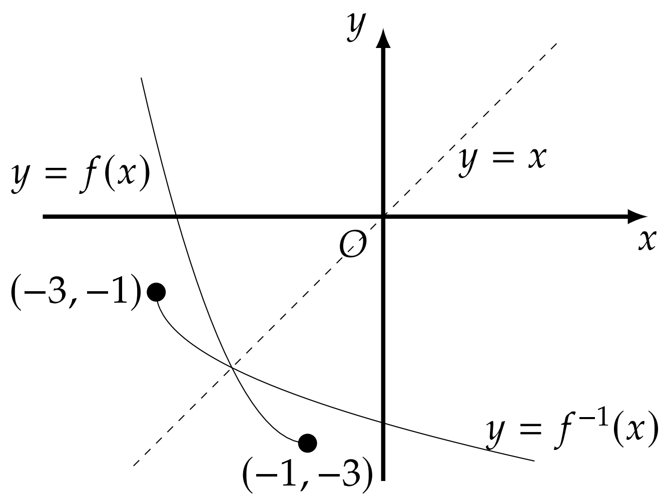
Remark:
Note that we have used $D_{f^{-1}} = R_f$ (see previous example).

1.4 Relationship between $y = f(x)$ and $y = f^{-1}(x)$

Example 1.4: Symmetry between $y = f(x)$ **and** $y = f^{-1}(x)$

The function f is defined by

$$f : x \mapsto x^2 + 2x - 2, \qquad x \in \mathbb{R}, x \leq -1.$$



The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are ***symmetrical*** about the line $y = x$. ■

The intersection of the two curves also coincide with $y = x$.

Hence the solution to $f(x) = f^{-1}(x)$ can be found by solving the equation $f(x) = x$.

2 Composite functions

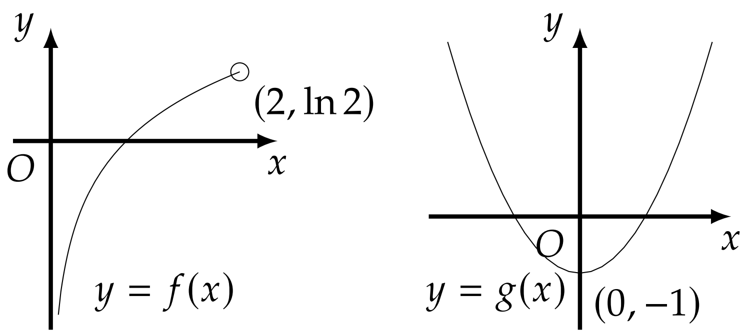
2.1 Existence of composite functions

- The composite function fg ***exists*** if $\boxed{R_g \subseteq D_f}$
- The composite function fg ***does not exist*** if $\boxed{R_g \not\subseteq D_f}$

Example 2.1: Existence of composite functions

The functions f and g are defined by

$$\begin{aligned} f &: x \mapsto \ln x, && x \in \mathbb{R},\ 0 < x < 2, \\ g &: x \mapsto x^2 - 1, && x \in \mathbb{R}. \end{aligned}$$



$R_f = (-\infty, \ln 2)$, $D_g = (-\infty, \infty)$.
Hence $R_f \subseteq D_g \Rightarrow$ the composite gf exists. ■

$R_g = [-1, \infty)$, $D_f = (0, 2)$.
Hence $R_g \not\subseteq D_f \Rightarrow$ the composite fg does not exist. ■

2.2 Domain and range of composite functions

- $D_{gf} = D_f$ $\qquad$ $D_{fg} = D_g$
- $D_f \xrightarrow{f} R_f \xrightarrow{g} R_{gf}$ $\qquad$ $D_g \xrightarrow{g} R_g \xrightarrow{f} R_{fg}$

2.2.1 Domain of composite function

Example 2.2A: Domain of a composite function

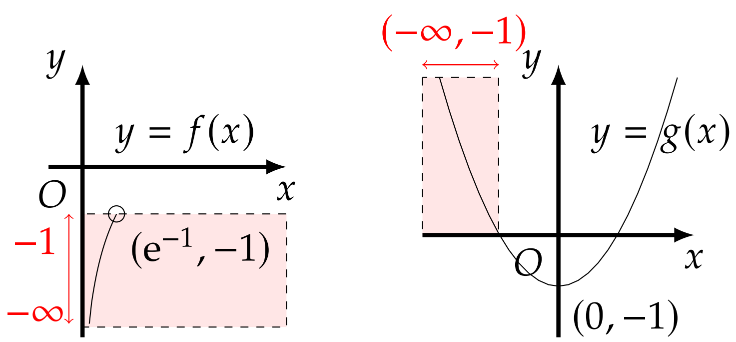
The functions f and g are defined by

$$f : x \mapsto \ln x, \qquad x \in \mathbb{R},\ 0 < x < \mathrm{e}^{-1},$$
$$g : x \mapsto x^2 - 1, \qquad x \in \mathbb{R}.$$

$D_{gf} = D_f = (0, \mathrm{e}^{-1})$. ■

2.2.2 Range of composite function

Example 2.2B: Range of a composite functions



$D_f = (0, \mathrm{e}^{-1}) \xrightarrow{f} R_f = (-\infty, -1) \xrightarrow{g} R_{gf} = (0, \infty)$.
$R_{gf} = (0, \infty)$. ■

2.3 Rule for composite functions

Example 2.3: Rule of composite functions

The functions f and g are defined by

$$f : x \mapsto -\frac{3}{4}x + 2, \qquad x \in \mathbb{R},\ 0 \le x \le 4,$$
$$g : x \mapsto x^2 - 1, \qquad x \in \mathbb{R},\ -1 \le x \le 2$$

$$\begin{aligned} fg(x) &= f\left(x^2 - 1\right) \\ &= -\frac{3}{4}\left(x^2 - 1\right) + 2 \\ &= -\frac{3}{4}x^2 + \frac{11}{4}. \ \blacksquare \end{aligned}$$

$$\begin{aligned} gf(x) &= g\left(-\frac{3}{4}x + 2\right) \\ &= \left(-\frac{3}{4}x + 2\right)^2 - 1. \ \blacksquare \end{aligned}$$

Definition of gf:

$$gf : x \mapsto \left(-\frac{3}{4}x + 2\right)^2 - 1, \qquad x \in \mathbb{R}, 0 \le x \le 4.$$

$\blacksquare$

Remark:
Note that we have used $D_{gf} = D_f$ (see previous example).

Links

- Math Atlas: notes and questions
  https://math-atlas.vercel.app
- Math Bounty: TYS answers and related questions
  https://math-bounty.vercel.app
- (Under construction)
  Math Interactive: mobile-friendly lecture notes
  https://math-interactive.vercel.app
- (Deprecated soon)
  AdotB: older website
  https://adotb.xyz

Contact me

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