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P_1 Synthetic Acceleration for Nonclassical Spectral $\mathrm{S_N}$ Equations in Slab Geometry

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1. INTRODUCTION

The theory of nonclassical particle transport was introduced by Larsen [1] to describe measurements of photon path-length in the Earth's cloudy atmosphere where distance-to-collision for photons is not exponentially distributed [2]. This nonclassical theory has since been extended and applied to different problems, such as neutron transport [3], computer graphics [4], and anomalous diffusion [5]. The nonclassical transport equation extends the phase-space to include an extra independent variable, the free-path s, that represents the distance traveled by a particle since its last interaction.

The slab geometry, one-speed nonclassical transport equation with isotropic scattering in its initial value form is [6]:

$$\frac{\partial}{\partial s}\Psi(x,\mu,s)+\mu\frac{\partial}{\partial x}\Psi(x,\mu,s)+\sigma_t(s)\Psi(x,\mu,s)=0, \tag{1a}$$

$$\Psi(x,\mu,0)\equiv\varphi(x)=\frac{c}{2}\int_{-1}^{1}d\mu'\int_0^\infty ds'\,\sigma_t(s')\Psi(x,\mu',s')+\frac{Q(x)}{2}, \tag{1b}$$

where x and μ represent position and scattering angle cosine, Ψ is the nonclassical angular flux, c is the scattering ratio, and Q is an isotropic source. Observe that $\Psi(x,\mu,0)$ as given in Eq. (1b) is a function of x only, represented by $\varphi(x)$. The total cross section σ_t is modeled as a function of the free-path s and is related to the particle free-path distribution by

$$p(s)=\sigma_t(s)e^{-\int_0^s ds'\sigma_t(s')}\,. \tag{2}$$

In the case of classical transport the s-dependence is removed; Eq. (2) then becomes the exponential $p(s)=\sigma_t e^{-\sigma_t s}$, and Eqs. (1) reduce to the classical steady-state, one-speed linear Boltzmann equation with isotropic scattering in slab geometry [6].

Until recently, deterministic numerical results for the nonclassical transport equation given by Eqs. (1) were only available for problems in rod geometry (cf. [7]). This is in part because the s-dependence of σ_t and the improper integral on Eq. (1b) tend to make an approach involving the discretization of the variable s

inefficient. As a first step to address this issue, a set of nonclassical spectral S_N equations has been recently introduced [8]. In this approach, the spectral method is used to represent the nonclassical flux as a series of Laguerre polynomials [9] in the variable s. The resulting nonclassical equation is then approximated by the S_N formulation and numerically solved by the conventional fine-mesh Diamond Difference (DD) method with a source iteration scheme.

For highly scattering systems, the spectral radius of transport problems can get arbitrarily close to unity [10], and acceleration becomes key to an efficient solver. The goal of the present work is to introduce a moment-based, P_1 synthetic acceleration (P_1SA) technique to accelerate the convergence of the nonclassical spectral S_N equations. In the full version of this work, we will provide more details about its effectiveness using numerical results. This will be the first time the solution of these equations will be accelerated.

The remainder of this summary is organized as follows. In Section 2, we sketch the derivation of the nonclassical spectral S_N equations and source iteration. In Section 3, we present the P_1SA technique. We conclude our summary in the final section with a brief discussion.

2. NONCLASSICAL SPECTRAL S_N AND SOURCE ITERATION

This section presents a brief sketch of the derivation of the nonclassical spectral S_N equations. For a detailed derivation, we direct the reader to the work presented in [8].

Defining $\hat{\psi}$ such that

$$\Psi(x,\mu,s) \equiv \hat{\psi}(x,\mu,s)e^{-\int_0^s ds'\sigma_t(s')} \tag{3}$$

and substituting it in Eqs. (1), we obtain the following nonclassical problem:

$$\frac{\partial}{\partial s}\hat{\psi}(x,\mu,s) + \mu\frac{\partial}{\partial x}\hat{\psi}(x,\mu,s) = 0, \tag{4a}$$

$$\hat{\psi}(x,\mu,0) \equiv \hat{\varphi}(x) = \frac{c}{2}\int_{-1}^{1} d\mu' \int_0^\infty ds' p(s')\hat{\psi}(x,\mu',s') + \frac{Q(x)}{2}\,, \tag{4b}$$

where $p(s')$ is the free-path distribution given in Eq. (2). Here, we assume vacuum boundaries. To apply the spectral method, we represent $\hat{\psi}$ using a Laguerre polynomial expansion [9]:

$$\hat{\psi}(x,\mu,s) = \sum_{m=0}^{M}\psi_m(x,\mu)L_m(s), \tag{5}$$

where M is the expansion order for Laguerre polynomials. We introduce this expansion in the nonclassical problem given by Eqs. (4). Then, the following steps take place (cf. [8]): (i) we multiply Eq. (4a) by $e^{-s}L_m(s)$; (ii) we integrate from 0 to ∞ with respect to s; and (iii) we perform algebraic manipulations to simplify the result. This procedure returns the nonclassical spectral equation

$$\mu\frac{\partial}{\partial x}\psi_m(x,\mu) + \psi_m(x,\mu) = \frac{c}{2}\int_{-1}^{1} d\mu' \int_0^\infty ds' p(s')\sum_{k=0}^{M}\psi_k(x,\mu')L_k(s') + \frac{Q(x)}{2} - \sum_{j=0}^{m-1}\psi_j(x,\mu), \tag{6a}$$

$$m = 0,1,...,M,$$

and the *classical* angular flux $\Psi_c(x,\mu) = \int_0^\infty ds \Psi(x,\mu,s)$ is given by

$$\Psi_c(x,\mu) = \sum_{m=0}^{M} \psi_m(x,\mu) \int_0^\infty ds\, L_m(s) e^{-\int_0^s ds' \sigma_t(s')} \,. \tag{6b}$$

Introducing the S_N formulation [10] in Eqs. (6), we obtain

$$\mu_n \frac{d}{dx}\psi_{m,n}(x) + \psi_{m,n}(x) = \frac{c}{2}\sum_{l=1}^{N}\omega_l \sum_{k=0}^{M}\psi_{k,l}(x)\int_0^\infty ds' p(s') L_k(s') + \frac{Q(x)}{2} - \sum_{j=0}^{m-1}\psi_{j,n}(x), \tag{7a}$$

$$m = 0,1,...,M, \quad n = 1,...,N,$$

$$\Psi_{c,n}(x) = \sum_{m=0}^{M}\psi_{m,n}(x)\int_0^\infty ds\, L_m(s) e^{-\int_0^s ds'\, \sigma_t(s')}, \quad n = 1,2,...,N \,. \tag{7b}$$

In order to solve the nonclassical spectral S_N equations using standard source iteration [10], we lag the scattering source and other terms on the right-hand side:

$$\mu_n \frac{d}{dx}\psi^{i+1}_{m,n}(x) + \psi^{i+1}_{m,n}(x) = \frac{c}{2}\sum_{l=1}^{N}\omega_l \sum_{k=0}^{M}\psi^{i}_{k,l}(x)\int_0^\infty ds' p(s') L_k(s') + \frac{Q(x)}{2} - \sum_{j=0}^{m-1}\psi^{i}_{j,n}(x), \tag{8}$$

where i is the iteration index.

3. P_1 SYNTHETIC ACCELERATION

Subtracting the exact nonclassical spectral S_N equation from Eq. (8) yields the following error equation:

$$\mu_n \frac{d}{dx}\epsilon^{i+1}_{m,n}(x) + \epsilon^{i+1}_{m,n}(x) = \frac{c}{2}\sum_{k=0}^{M}\upsilon^{i}_{k}(x)\int_0^\infty ds' p(s') L_k(s') - \sum_{j=0}^{m-1}\epsilon^{i}_{j,n}(x), \tag{9}$$

where $\epsilon^{i+1}_{m,n}(x) = \psi_{m,n}(x) - \psi^{i+1}_{m,n}(x)$ is the error in the m^{th} Laguerre moment of angular flux along direction μ_n. Here, $\upsilon_k(x) = \sum_{l=1}^{N}\omega_l \epsilon_{k,l}(x)$ is the zeroth Legendre moment of $\epsilon_{k,n}(x)$.

For highly scattering problems, acceleration is important for an efficient solution due to the spectral radius becoming arbitrarily close to unity [10]. In this work, we propose P_1 acceleration, where the error equation above is represented using the following P_1 equations:

$$\frac{d}{dx}\Upsilon_m(x) + \upsilon_m(x) = \frac{c}{2}\sum_{k=0}^{M}\upsilon_k(x)\int_0^\infty ds' p(s') L_k(s') - \sum_{j=0}^{m-1}\upsilon_j(x) \quad m = 0,1,...M, \tag{10a}$$

$$\frac{1}{3}\frac{d}{dx}\upsilon_m(x) + \Upsilon_m(x) = -\sum_{j=0}^{m-1}\Upsilon_j(x) \quad m = 0,1,...M \,. \tag{10b}$$

Here, $\Upsilon_k(x) = \sum_{l=1}^{N}\mu_l\omega_l\epsilon_{k,l}(x)$ is the first Legendre moment of $\epsilon_{k,n}(x)$. This set of $2\times(M+1)$ equations is used in the error correction stage [10], which results in the P_1 synthetic acceleration scheme.

4. DISCUSSION

A set of nonclassical spectral S_N equations has been recently introduced as an option to provide deterministic numerical results of the nonclassical transport equation [8]. These equations are obtained through a spectral method that represents the nonclassical flux as a series of Laguerre polynomials in the variable s. The resulting nonclassical equation has a classical form, and is then solved using the S_N formulation with the DD method and a source iteration scheme.

It is known that the spectral radius of transport problems can get arbitrarily close to unity in highly scattering systems [10]. For this reason, the goal of the present work is to introduce a moment-based, P_1 synthetic acceleration (P_1SA) technique to accelerate the convergence of the nonclassical spectral S_N equations. In this summary we have presented a sketch of the derivation of the nonclassical spectral S_N equations and have discussed the P_1SA approach. In the full version of this work, we will validate the effectiveness of this scheme using numerical results. This will be the first time the solution of these equations will be accelerated.

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