*The 26th International Conference on Transport Theory (ICTT-26)*
Paris, France, September 23-27, 2019

# Asymptotic Derivation of the Simplified $\mathbf{P}_N$ Equations for Nonclassical Transport with Anisotropic Scattering

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## 1. INTRODUCTION

An accurate model of particle transport through scattering and absorbing media is necessary for the understanding of many phenomena in nuclear engineering and physics. Classical particle transport assumes that, in spatially homogeneous media, the distribution of the particle's free-path length $s$ (distance traveled since the particle's previous interaction) is given by the exponential

$$p(s) = \Sigma_t e^{-\Sigma_t s}, \tag{1}$$

where $\Sigma_t$ is the total cross section. This distribution is valid if the locations of the scattering centers are uncorrelated (Poisson distributed).

However, in certain inhomogeneous media the positions of the scattering centers may be spatially correlated, resulting in a flux attenuation that is not exponential in nature. Examples of such systems include neutron transport in pebble bed reactor (PBR) cores [1]; the scattering of solar radiation through clouds [2]; and the scattering of light through a glass matrix embedded with high-refractive index particles [3]. To address such cases, the theory of *nonclassical* particle transport was developed.

In nonclassical particle transport, the total cross section $\Sigma_t$ is no longer treated as independent of the free-path. Instead, it is modeled as a function of the free-path length $s$. The $s$-dependent total cross section is related to the particle free-path distribution by

$$p(s) = \Sigma_t(s) e^{-\int_0^s \Sigma_t(s')ds'}. \tag{2}$$

We observe that, if the total cross section is a constant with respect to $s$, Eq. (2) reduces to the exponential distribution given by Eq. (1).

The nonclassical transport equation [4] consists of a linear Boltzmann equation on an extended phase space, able to model particle transport for any given free-path distribution. In its steady-state, monoenergetic form, it is written as

$$\frac{\partial}{\partial s}\Psi(\boldsymbol{x},\boldsymbol{\Omega},s)+\boldsymbol{\Omega}\cdot\nabla\Psi(\boldsymbol{x},\boldsymbol{\Omega},s)+\Sigma_t(s)\Psi(\boldsymbol{x},\boldsymbol{\Omega},s)=\delta(s)\left[\int_{4\pi}\int_0^\infty c\Sigma_t(s)P(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')\Psi(\boldsymbol{x},\boldsymbol{\Omega}',s')ds'd\Omega'+\frac{Q(\boldsymbol{x})}{4\pi}\right]. \quad (3)$$

Here, $\Psi(\boldsymbol{x},\boldsymbol{\Omega},s)$ is the nonclassical angular flux, $c$ is the scattering ratio (probability of scattering), and $Q$ is an isotropic source. The $\delta(s)$ operator in the right-hand side is necessary because the scattering (or creation) of a particle resets the traveled free-path $s$ to zero. If the $s$-dependence is removed, Eq. (3) reduces to the classical steady-state, monoenergetic Boltzmann transport equation. The classical angular flux $\psi(\boldsymbol{x},\boldsymbol{\Omega})$ can be recovered from the solution of Eq. (3) by integrating over the free-path $s$:

$$\psi(\boldsymbol{x},\boldsymbol{\Omega})=\int_0^\infty\Psi(\boldsymbol{x},\boldsymbol{\Omega},s)ds\,. \quad (4)$$

The simplified $P_N$ ($SP_N$) equations are a high-order asymptotic approximation of the classical transport equation (cf. [5]). Recently, a set of nonclassical $SP_N$ equations was derived [6] for the particular case of nonclassical transport with isotropic scattering, in which $P(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')=1/4\pi$. In this work, we use an asymptotic analysis to generalize this result to include anisotropic scattering. We will verify that the resulting set of equations reduces to the classical $SP_N$ equations with anisotropic scattering when $p(s)$ is given by Eq. (1), thus showing that the hierarchy of $SP_N$ equations can be generalized for nonclassical transport theory.

The remainder of this summary is organized as follows. In Section 2 we present a sketch of the asymptotic analysis and the explicit equation for nonclassical $SP_1$ (nonclassical diffusion). We close with a short discussion in Section 3, in which we describe what will be presented in the full version of this work.

## 2. SKETCH OF THE ASYMPTOTIC ANALYSIS

Defining $0<\varepsilon\ll 1$, the following scaling relationships are applied, assuming a diffusive environment in which scattering is much larger than particle absorption and production:

$$\Sigma_t(s)=\frac{\sigma(s/\varepsilon)}{\varepsilon}\,,\qquad Q(\boldsymbol{x})=\varepsilon q(\boldsymbol{x})\,,\qquad 1-c=\varepsilon^2\kappa\,. \quad (5a)$$

Here, $\sigma,\kappa$, and $q$ are $O(1)$. This scaling is consistent with the one used in [6] and results in the scaled moments

$$\langle s^m\rangle=\varepsilon^m\int_0^\infty\left(\frac{s}{\varepsilon}\right)^m\frac{\sigma(s/\varepsilon)}{\varepsilon}e^{-\int_0^s\frac{\sigma(s'/\varepsilon)}{\varepsilon}ds'}ds=\varepsilon^m\langle s^m\rangle_\varepsilon, \quad (5b)$$

where $\langle s^m\rangle_\varepsilon$ is $O(1)$.

Next, we define

$$\Psi(\boldsymbol{x},\boldsymbol{\Omega},\varepsilon s)\equiv\Psi_\varepsilon(\boldsymbol{x},\boldsymbol{\Omega},s) \quad (6a)$$

and

$$\Psi_\varepsilon(\boldsymbol{x},\boldsymbol{\Omega},s) \equiv \hat{\Psi}(\boldsymbol{x},\boldsymbol{\Omega},s)\frac{e^{-\int_0^\infty \frac{\sigma(s'/\varepsilon)}{\varepsilon}ds'}}{\varepsilon\langle s\rangle_\varepsilon}\,. \tag{6b}$$

Employing the scaling relationships given by Eqs. (5) and the definitions given by Eqs. (6), Eq. (3) can now be transformed into the following scaled nonclassical Boltzmann transport equation:

$$\hat{\Psi}(\boldsymbol{x},\boldsymbol{\Omega},s) + \varepsilon\boldsymbol{\Omega}\cdot\nabla\int_0^s \hat{\Psi}(\boldsymbol{x},\boldsymbol{\Omega},s')ds' = \int_{4\pi}\int_0^\infty (1-\varepsilon^2\kappa)p(s')P(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')\hat{\Psi}(\boldsymbol{x},\boldsymbol{\Omega}',s')ds'd\Omega' + \varepsilon^2\langle s\rangle_\varepsilon\frac{q(\boldsymbol{x})}{4\pi}. \tag{7}$$

We let the nonclassical scalar flux $\varphi$ be given by

$$\varphi(\boldsymbol{x},s) = \int_{4\pi}\hat{\Psi}(\boldsymbol{x},\boldsymbol{\Omega}',s)d\Omega', \tag{8}$$

and define the operator $\mathcal{J}$ such that

$$\mathcal{J}\hat{\Psi}(\boldsymbol{x},\boldsymbol{\Omega},s) \equiv \frac{1}{4\pi}\int_{4\pi}\hat{\Psi}(\boldsymbol{x},\boldsymbol{\Omega}',s)d\Omega'. \tag{9}$$

We also represent $P(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')$, the distribution of particles with direction of flight $\boldsymbol{\Omega}'$ that scatter into direction of flight $\boldsymbol{\Omega}$, using the following Legendre polynomial expansion:

$$P(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}') = \sum_{m=0}^\infty \frac{2m+1}{4\pi}c_m P_m(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}'), \tag{10}$$

where $P_m(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')$ is the $m$-th order Legendre polynomial, $c_0 = 1$, and $c_1 = \bar{\mu}_0$ (the mean scattering cosine).

Next, we derive a function to generate $\hat{\Psi}(\boldsymbol{x},\boldsymbol{\Omega},s)$ asymptotically. We operate on Eq. (7) using Eqs. (8) to (10), and use a Taylor expansion about $\varepsilon = 0$ to yield the following generating function

$$\hat{\Psi}(\boldsymbol{x},\boldsymbol{\Omega},s) = \frac{1}{4\pi}\sum_{n=0}^\infty(-1)^n\varepsilon^n\left[\jmath(I-\mathcal{J})\boldsymbol{\Omega}\cdot\nabla\int_0^s(\cdot)ds'\right]^n\varphi(\boldsymbol{x},s), \tag{11a}$$

where

$$\jmath\hat{\Psi}(\boldsymbol{x},\boldsymbol{\Omega},s) = \hat{\Psi}(\boldsymbol{x},\boldsymbol{\Omega},s) + \int_{4\pi}\int_0^\infty(1-\varepsilon^2\kappa)p(s')\sum_{m=1}^\infty\frac{2m+1}{4\pi}c_m P_m(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')\hat{\Psi}(\boldsymbol{x},\boldsymbol{\Omega},s)ds'd\Omega'. \tag{11b}$$

Finally, we evaluate Eq. (11a) for $n = 0$ and $n = 1$ and truncate the results for terms of order greater than $O(\varepsilon^2)$. The parameters in the resulting equation are rescaled according to Eqs. (5), yielding the anisotropic nonclassical SP$_1$ equation [4]:

$$-\frac{1}{3}\left[\frac{\langle s^2\rangle}{2\langle s\rangle} + \frac{c\bar{\mu}_0}{1-c\bar{\mu}_0}\langle s\rangle\right]\nabla^2\Phi(\boldsymbol{x}) + \frac{1-c}{\langle s\rangle}\Phi(\boldsymbol{x}) = Q(\boldsymbol{x}), \tag{12}$$

where $\Phi(\boldsymbol{x}) = \int_{4\pi}\int_0^\infty \Psi(\boldsymbol{x},\boldsymbol{\Omega},s)dsd\Omega$ is the classical scalar flux. This is the first of the anisotropic nonclassical SP$_N$ equations. We remark that, if the free-path distribution is given by Eq. (1), then Eq. (12) reduces to the classical diffusion equation with anisotropic scattering

$$-\frac{1}{\Sigma_t(1-c\bar{\mu}_0)}\nabla^2\Phi(\boldsymbol{x}) + \Sigma_a\Phi(\boldsymbol{x}) = Q(\boldsymbol{x})\,. \tag{13}$$

## 3. DISCUSSION

Classical particle transport theory assumes that the particle flux is attenuated exponentially. However, there are transport regimes in which the distribution of particle free-path lengths is nonexponential. Nonclassical transport theory addresses this problem by making the particle angular flux dependent upon the free-path $s$ in order to preserve the nonclassical nature of particle transport.

The SP$_N$ equations have been successful at solving the classical transport equation in diffusive regimes. In this summary, we discuss for the first time the derivation of nonclassical SP$_N$ equations to approximate nonclassical transport with *anisotropic* scattering, generalizing the previous result presented in [6]. This derivation is done through an asymptotic analysis; a preliminary sketch of this analysis is presented in Section 2, in which the SP$_1$ (diffusion) equation has been determined in agreement with previous results [4].

The full version of this work will present–for the first time–explicit equations for nonclassical SP$_2$ and SP$_3$ with anisotropic scattering, as well as a procedure to obtain higher order SP$_N$ equations. We will show that the resulting set of equations reduces to the classical SP$_N$ equations under the assumption of classical transport. This result will expand our current understanding of the diffusive behavior of the nonclassical transport equation.

## ACKNOWLEDGMENTS

The authors acknowledge support under award number NRC-HQ-84-15-G-0024 from the Nuclear Regulatory Commission. The statements, findings, conclusions, and recommendations are those of the authors and do not necessarily reflect the view of the U.S. Nuclear Regulatory Commission.

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