# Exact Transport Representations of the Classical and Nonclassical Simplified $\mathrm{P}_N$ Equations

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## 1. Introduction

A nonclassical linear Boltzmann equation has been recently proposed [1] to address transport problems in which the particle flux is not attenuated exponentially. This theory requires the use of a "memory variable", namely the free-path $s$, representing the distance traveled by a particle since its previous interaction. Assuming that scattering is isotropic, the one-speed nonclassical transport equation with an isotropic internal source is written as [1]

$$\frac{\partial\psi}{\partial s} + \mathbf{\Omega}\cdot\nabla\psi + \Sigma_t(s)\psi = \frac{\delta(s)}{4\pi}\left[c\int_{4\pi}\int_0^\infty \Sigma_t(s')\psi(\mathbf{x},\mathbf{\Omega}',s')ds'd\Omega' + Q(\mathbf{x})\right]. \tag{1}$$

Here, $\psi = \psi(\mathbf{x},\mathbf{\Omega},s)$ represents the nonclassical angular flux, $c$ is the scattering ratio, and $Q(\mathbf{x})$ is the source. The total cross section $\Sigma_t$ is a function of $s$ such that the free-path probability distribution function $p(s) = \Sigma_t(s)e^{-\int_0^s \Sigma_t(s')ds'}$ does not have to be exponential.

It has been shown that certain cases in the hierarchy of the classical simplified $\mathrm{P}_N$ equations ($\mathrm{SP}_1$, $\mathrm{SP}_2$, and $\mathrm{SP}_3$) can be represented exactly by a nonclassical transport equation [2]. This has also been shown for nonclassical diffusion [3]. In particular, these results yield explicit expressions for the free-path distribution $p(s)$ that make it possible to consistently simulate diffusion, $\mathrm{SP}_2$, and $\mathrm{SP}_3$ problems using a Monte Carlo method.

In the theoretical portion of this work, we show that this result can be extended for the set of *nonclassical* simplified $\mathrm{P}_N$ equations recently introduced in [4]. We find $p(s)$ and the corresponding $\Sigma_t(s)$ such that the integral forms of these nonclassical $\mathrm{SP}_N$ equations are identical to the integral equation for Eq. (1).

Moreover, we present for the first time numerical simulations that validate the theory proposed here and in references [2,3]. Specifically, we perform Monte Carlo simulations in which the free-paths are sampled from the appropriate nonexponential distributions and demonstrate that they match the solutions obtained with both the classical and the nonclassical forms of the $\mathrm{SP}_N$ equations. This effectively shows that it is possible to solve diffusion, $\mathrm{SP}_2$, and $\mathrm{SP}_3$ problems using a nonclassical Monte Carlo *transport* method.

## 2. Numerical Results

The free-path distribution functions for the classical $\mathrm{SP}_N$ equations are given by [2]

$$p(s) = 3\Sigma_t^2 s e^{-\sqrt{3}\Sigma_t s} \tag{2}$$

for classical diffusion ($\mathrm{SP}_1$);

$$p(s) = \frac{25}{27}\Sigma_t^2 s e^{-\sqrt{5/3}\Sigma_t s} + \frac{4}{9}\delta(s) \tag{3}$$

for classical SP$_2$; and

$$p(s) = \Sigma_t^2 s \left[5.642025e^{-2.941340\Sigma_t s} + 0.469086e^{-1.161256\Sigma_t s}\right] \tag{4}$$

for classical SP$_3$. We consider slab geometry transport taking place in an one-dimensional (1-D) system such that $-50 \le x \le 50$. We assume vacuum boundaries at $x = \pm 50$ and define the source $Q(x)$ as

$$Q(x) = \begin{cases} 1, & \text{if} -0.5 \le x \le 0.5 \\ 0, & \text{otherwise} \end{cases} . \tag{5}$$

We perform Monte Carlo simulations in this system by sampling the free-path $s$ from the distributions given in Eqs. (2) to (4). Figure 1 shows the Monte Carlo estimated (MC SPN) scalar flux at $x = 0$ for different choices of the scattering ratio $c$. We compare these estimates with those obtained by directly solving the classical SP$_N$ equations (SPN) in this system, demonstrating that the Monte Carlo approach effectively reproduces the solution of the SP$_N$ equations.

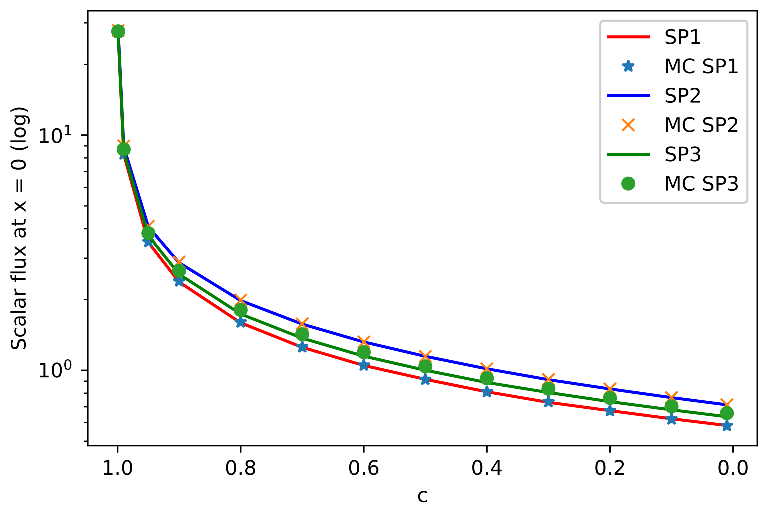


Figure 1: Scalar flux (log) at $x = 0$ for different choices of scattering ratio $c$

The full version of this work will contain the derivation for the explicit forms of the free-path distribution $p(s)$ for the nonclassical SP$_N$ equations introduced in [4], demonstrating that they reduce to the ones in Eqs. (2) to (4) under the right (classical) assumptions. We will also present a full set of numerical solutions for the nonclassical SP$_N$ equations in a 1-D random periodic system, validating the theoretical predictions.

## References

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