Topic entire + Bdd = constant   Date

Entire Function: A complex function is said to be entire (sometimes integral) if it is analytic in the whole complex plane.

$P(z)$ = diff, cts, limit

e.g. (i) The polynomial

$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \; a_n \neq 0$

(ii) $f(z) = e^z = e^x(\cos y + i \sin y)$, are entire functions.

Singular Point or Singularity: If a function $f(z)$ fails to be analytic at a point z_0 but in every nbd of z_0 there exist atleast one point where the function is analytic, then z_0 is said to be a singular point or singularity of $f(z)$.

e.g. (i) The function $\boxed{f(z) = \frac{1}{z}}$ is analytic except at $\boxed{z=0}$, and each deleted nbd $f(z)$ is analytic, so by definition $z=0$ is a singular point.

(ii) For $f(z) = \dfrac{z^3+7}{(z^2-2z+2)(z-3)}$

$z = 3, \; z = 1+i, \; z = 1-i$ are singularities.

$\frac{f(z)}{g(z)}$ $\quad \frac{z^2}{z}$ $\quad \frac{z}{z^2}$ $\quad \frac{\sin z}{z}$

Result: If $f(z)$ is nowhere analytic in D, then it has no singularity.

e.g. $f(z) = |z|$ has no singularity. (As nowhere analytic)

Topic Date

+P

C - Theorem : If $f(z) = u + iv$ be analytic in a domain and any one of the following condition satisfies, then $f(z)$ is a constant.

proof

MSQ

C-R eqn

(a) Either of u and v is constant

u = constant

$u + iv \to$ analytic

C-R eqn

$u_x = v_y$
$u_y = -v_x$

(b) $|f(z)|$ is constant

(c) If argument of $f(z)$ is constant

(d) If $u^2 = v$

proof

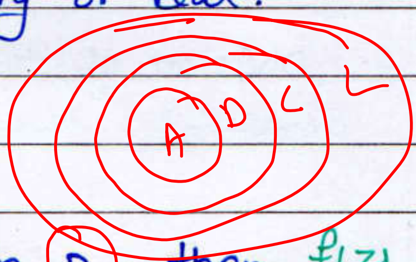
(e) If $v^2 = u$

(f) $f'(z)$ vanishes identically

(g) If u and v satisfies the equation $\boxed{au + bv = c}$, where a, b, c are non zero and $a^2 + b^2 \neq 0$.

(h) If u and v lies on a unit circle in w-plane i.e. $\boxed{u^2 + v^2 = 1}$

(i) If $f(z)$ is purely imaginary or real.



Important Results :

1. If $f(z)$ is analytic in domain D, then $f(z)$ is continuous and differentiable both in D.

$f(z) = u + iv$

2. Real and imaginary parts of an analytic function are separately continuous and differentiable in D.

3. (a) $\cosh^2 z - \sinh^2 z = 1$	4. (a) $\sin iz = i \sinh z$
(b) $\operatorname{sech}^2 z + \tanh^2 z = 1$	(b) $\cos iz = \cosh z$
(c) $\coth^2 z - \operatorname{cosech}^2 z = 1$	(c) $\tan iz = i \tanh z$

D D' D' C

D C

e^z → Analytic

D $\sin z$ But Not entire

D C D

In general C

$|z|<1$

— entire

Question →

C

entire bdd = Constant

Topic ENTIRE FUNCTION (IMPORTANT FACTS) Date

1. The range of a non-constant entire function is either $\mathbb{C}$ or $\mathbb{C}-\{a\}$.
i.e. Non-constant entire function may skip atmost one value from its codomain.

$e^z \rightarrow$ Range set

eg. (i) $f(z) = e^z$ skip $\{0\}$. $e^z = \cosh z + \sinh z$

★ If skip more than one value, then f is necessarily a constant.

$w = e^z$

$z = \log w$

(ii) constant (iii) Polynomial (iv) $\sin z$ (v) $\cos z$
(vi) $\sinh z$ (vii) $\cosh z$ are entire functions on 'C'.

2. Every entire bounded function is constant.
i.e. Entire + Bounded $\Longrightarrow$ Constant.

entire, Algebra $\frac{f}{g}$ $\frac{e^z}{\sin z}$ $D / \{ - \int$

3. Product of two entire function is again entire.

$\sin z = 0$

Ana

4. Division of two entire function is analytic function except at those points where denominator vanishes.

e.g. (i) $\tan z$ and $\sec z$ are analytic (continuous & diff^n) at $\mathbb{C}-\{(2n+1)\cdot\frac{\pi}{2}, n\in I\}$ $\{$As $\cos z = 0 \Rightarrow z = (2n+1)\cdot\frac{\pi}{2}, n\in I\}$

zeros

$\frac{\sin z}{\cos z}$

(ii) $\tanh z$ and $\operatorname{sech} z$ are analytic (continuous and diff^n) at $\mathbb{C}-\{(2n+1)\cdot\frac{\pi}{2}i, n\in I\}$.

$\tan x = \frac{\sin x}{\cos x}$ zero

5. Let $f(z)$ is entire, then $g(z) = \overline{f(z)}$ is entire iff $f(z)$ is constant.
e.g. For $f(z) = z$, $\overline{f(z)} = \bar{z}$ is nowhere analytic.

Topic________ Date________

Harmonic Functions : A real function h of two variables is said to be harmonic in a domain D, if

(a) It has second order continuous partial derives

(b) Satisfies Laplace equation

$\frac{1}{x}$ (0,1) Polyno

$$\boxed{h_{xx} + h_{yy} = 0}$$ PDE

e.g. $h(x,y) = 3x^2y - y^3 + 4$ is harmonic in $\mathbb{C}$.

Since, $h_x = 6xy \Rightarrow \boxed{h_{xx} = 6y}$

$h_y = -3y^2 \Rightarrow \boxed{h_{yy} = -6y}$

$\Rightarrow h_{xx} + h_{yy} = 0$, obviously, the second order partial derivatives of $h(x,y)$ are continuous.

Thus, $h(x,y)$ is harmonic.

Result : If a function $f(z) = u + iv$ is analytic in a domain D, then its component functions u and v are harmonic in D.

(Converse not true).

Harmonic

$\sim$ Harmonic

* Harmonic + C-R = Analytic

$\sim$CR $\Rightarrow$ $\sim$ Analytic $\Rightarrow$ $\sim$ Analytic

Harmonic Conjugate : The function q is said to be a harmonic conjugate of p u if

(a) p and q are harmonic $\boxed{f(z) = p + iq}$ fun

(b) First order partial derivatives of p and q satisfy the C-R equations. $\boxed{p_x = q_y}$ $f(z) = u + iv$ $\boxed{p_y = -q_x}$

i.e. Harmonic + C-R = Harmonic conjugate = Analytic

Topic: Show Date

Result: A function $f(z) = u + iv$ is analytic in D $\Longleftrightarrow$ v is harmonic conjugate of u.

iff

$\Rightarrow$ $u_{xx} + u_{yy} = v_{xx} + v_{yy} =$

Note: We can not take random harmonic functions u and v.

e.g. (a) Let $u = x$, $v = -y$ are harmonic

but $f(z) = u + iv = x - iy$

CR fail $\Rightarrow f(z) = \bar{z}$ (Nowhere analytic)

(b) For $g(z) = v + iu$

$g(z) = -y + ix = +i(x+iy) = iz$

pass $\Rightarrow g(z) = iz$ (which is analytic)

Ex

Result: Two functions $u(x,y)$ and $v(x,y)$ are harmonic conjugate of each other D iff they are constant. CR fuel

Note: Laplace eq^n to be satisfied not just for a set of points, but for an open set or a domain or simple connected domain.

infinite

e.g. $u = x^3 - y^3$, $u_{xx} + u_{yy} = 6(x-y)$
$= 0$ only at $y = x$.

The set $\{z = x + iy : y = x\}$ is not open disc, so we can not treat it as harmonic in open subset of C.

$| \; | = 0$
$\neq 0$

Topic ____ Date ____

Complex Form of Laplace Equation :

$z = x + iy$

$\bar{z} = x - iy$

$x = \frac{z+\bar{z}}{2}$

We have, $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

Let $u(x,y)$ is harmonic function, then

$$u_{xx} + u_{yy} = 0 \quad \text{——— (1)}$$

$u(x,y)$

Now, $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial z}$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{1}{2}u_x + \frac{1}{2i}u_y = \frac{1}{2}(u_x - iu_y)$$

$$\frac{\partial^2 u}{\partial \bar{z}\,\partial z} = \frac{1}{2}\frac{\partial}{\partial x}(u_x - iu_y)\cdot\frac{\partial x}{\partial \bar{z}} + \frac{1}{2}\frac{\partial}{\partial y}(u_x - iu_y)\cdot\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{1}{2}u_{xx}\cdot\frac{1}{2} + \frac{1}{2}(-iu_{yy})\cdot\left(-\frac{1}{2i}\right)$$

$$= \frac{1}{4}(u_{xx} + u_{yy}) = 0 \quad \{\text{using (1)}\}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial \bar{z}\,\partial z} = 0}$$

This is required complex form of Laplace equation.

★ From preceeding discussion, the Laplacian operator has the equivalent form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial^2 u}{\partial \bar{z}\cdot\partial z}$$

Result

or,

$$\boxed{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\cdot\frac{\partial^2}{\partial \bar{z}\,\partial z}}$$

Topic Date

Ex. Let $f(z)$ be an analytic function, then prove that

(a) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4\,|f'(z)|^2$ MSE

(b) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\text{Re } f(z)|^2 = 2\,|f'(z)|^2$ Crate

Solution: (a) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \cdot |f(z)|^2 = 4 \cdot \frac{\partial^2}{\partial \bar{z} \cdot \partial z} [f(z) \cdot \overline{f(z)}]$

$$= 4 \cdot \frac{\partial^2}{\partial \bar{z} \cdot \partial z} [f(z) \cdot \bar{f}(\bar{z})]$$

$$= 4 \cdot \frac{\partial}{\partial z} [f(z) \cdot \bar{f}'(\bar{z})]$$

$$= 4\, f'(z) \cdot \overline{f'(z)} = 4\,|f'(z)|^2$$

(b) can be verified by using $\text{Re } f(z) = \frac{f(z) + \overline{f(z)}}{2}$

General Form :

(a) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^n = n^2\, |f(z)|^{n-2}\, |f'(z)|^2$

(b) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\text{Re } f(z)|^n = n(n-1)\, |f(z)|^{n-1}\, |f(z)|^2$

Ex

State

Topic: CONSTRUCTION OF AN HARMONIC FUNCTION

Proposition: Let $u(x,y)$ be harmonic in some nbd of the point (x_0, y_0). Then there exists a conjugate harmonic function $v(x,y)$ defined in the nbd and $f(z) = u(x,y) + i\,v(x,y)$ is an analytic function.

Formula for obtaining Analytic Function $f(z)$:

(a) when real part of $f(z)$ is given

Let $f(z) = u(x,y) + i\,v(x,y)$ — ①

$\overline{f(z)} = u(x,y) - i\,v(x,y)$ — ②

$$u(x,y) = \frac{1}{2}\left[f(z) + \overline{f(z)}\right]$$

$$u(x,y) = \frac{1}{2}\left[f(x+iy) + \bar{f}(x-iy)\right] \quad \text{— ③}$$

$\left\{\because \overline{f(x+iy)} = \bar{f}(x-iy), \text{ and } \overline{f(z)} = \bar{f}(\bar{z})\right\}$

$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{1}{2}\left[f\left(\frac{z}{2} + i\frac{z}{2i}\right) + \bar{f}\left(\frac{z}{2} - i\frac{z}{2i}\right)\right]$$

$$\Rightarrow f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - \bar{f}(0) \quad \text{— ④}$$

From ③, $u(0,0) = \frac{1}{2}\left[f(0) + \overline{f}(0)\right]$ — ⑤

From ②, $\overline{f(0)} = \text{Re } f(0) - i\,\text{Im } f(0)$

$\overline{f(0)} = u(0,0) - iC$

From ⑤, $$\boxed{f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + iC}$$

Topic Date

Ex. $u = 2xy + 2x$, Find $f(z)$.

Solution: $\because$ $f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ic$

$\therefore$ $f(z) = 2\left[\cancel{2}\cdot\frac{z}{\cancel{2}}\cdot\frac{z}{\cancel{2}i} + \cancel{2}\cdot\frac{z}{\cancel{2}}\right] - 0 + ic$

$\Rightarrow$ $f(z) = \frac{z^2}{i} + 2z + ic$

$\Rightarrow$ $\boxed{f(z) = -iz^2 + 2z + ic}$

(b) when imaginary part of $f(z)$ is given

Subtracting ① from ②, we get

$2i\,v(x,y) = f(z) - \overline{f(z)}$

Let $x = \frac{z}{2}$ and $y = \frac{z}{2i}$, we have

$2i\,v\left(\frac{z}{2}, \frac{z}{2i}\right) = f(z) - \overline{f}(0)$

$\Rightarrow$ $f(z) = 2i\,v\left(\frac{z}{2}, \frac{z}{2i}\right) + \overline{f}(0)$ —⑥

Now, $\overline{f}(z) = \text{Re}\, f(z) - i\, \text{Im}\, f(z)$

$\overline{f(0)} = \text{Re}\, f(0) - i\, \text{Im}\, f(0)$

$\Rightarrow$ $\overline{f}(0) = c - i\,v(0,0)$ —⑦

using this in ⑥, we get

$$\boxed{f(z) = 2i\,v\left(\frac{z}{2}, \frac{z}{2i}\right) - i\,v(0,0) + c}$$

Topic CONSTRUCTION OF ANALYTIC FUNCTION Date 26.12.21

I. Milne - Thomson Method : (For constructing analytic functions when the real or imaginary component is given and hence finding harmonic conjugates).

(a) when $u(x,y)$ is given :

We have, $z = x+iy$ so that $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

$$w = f(z) = u + iv = u(x,y) + i\,v(x,y)$$

or,

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}\right)$$

Setting $x = z$ and $y = 0 \Rightarrow z = \bar{z}$

$$f(z) = u(z,0) + i\,v(z,0)$$

$$f'(z) = \frac{dw}{dz} = \frac{dw}{dx} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

(Using C-R equations)

Let $\frac{\partial u}{\partial x} = \phi_1(x,y) = \phi_1(z,0)$

$\frac{\partial u}{\partial y} = \phi_2(x,y) = \phi_2(z,0)$

$f'(z) = \phi_1(z,0) - i\,\phi_2(z,0)$, on integregration,

$$\boxed{f(z) = \int \{\phi_1(z,0) - i\,\phi_2(z,0)\}\cdot dz + c}$$

(b) when $v(x,y)$ is given :

$$\boxed{f(z) = \int \{\psi_1(z,0) + i\,\psi_2(z,0)\}\cdot dz + c'}$$

where, $\psi_1 = \frac{\partial v}{\partial y}$ and $\psi_2 = \frac{\partial v}{\partial x}$.

Topic ________ Date ________

Ex. $u(x,y) = 2xy + 2x$, Find $f(z)$.

Solution: $\phi_1(z,0) = \left.\frac{\partial u}{\partial x}\right|_{(z,0)} = (2y+2)\Big|_{(z,0)} = 2$

$\phi_2(z,0) = \left.\frac{\partial u}{\partial y}\right|_{(z,0)} = 2x\Big|_{(z,0)} = 2z$

Now, $f(z) = \int \{\phi(z,0) - i\phi_2(z,0)\} \cdot dz + c$

$\Rightarrow f(z) = \int (2 - 2iz) \cdot dz + c$

$$\boxed{f(z) = 2z - iz^2 + c}$$

★ Without Method : $\because f(z) = u + iv$

$\therefore f'(z) = u_x + i v_x$

$\Rightarrow f'(z) = u_x - i u_y$ —①

$\{\because u_y = -v_x\}$

Now, $u = 2xy + 2x$

$u_x = 2y + 2$ and $u_y = 2x$

Using these values in ①, we get

$f'(z) = 2y + 2 - i(2x) = -2i(x+iy) + 2$

$\Rightarrow f'(z) = -2iz + 2$, on Integration,

$$\boxed{f(z) = -iz^2 + 2z + c}$$

Topic ______ Date ______

II. Method : (Suppose $f(z) = u + iv$ is analytic and $u(x,y)$ is Known) $v(x,y)$ can be determined.

$$\because \quad dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy$$

$$dv = \left(-\frac{\partial u}{\partial y}\right) \cdot dx + \left(\frac{\partial u}{\partial x}\right) \cdot dy \quad \text{(By C-R eq}^n\text{)}$$

Taking $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$

i.e. $dv = M\,dx + N\,dy$ —— ①

$$\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

$$\therefore \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\nabla^2 u = 0$$

(As u satisfies Laplace equation)

$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0 \quad \text{or} \quad \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$\Rightarrow$ eq^n ① is exact diff^r equation.

So that, $v(x,y)$ can be determined by integrating equation ①.

Hence, $f(z)$ can be determined from the equation, $f(z) = u + iv$.

Note: while integrating, treating y as constant in M and Take terms which do not contain x in N.

Topic ______ Date ______

Ex. Find the analytic function of which the real part is $u(x,y) = e^x(x\cos y - y\sin y)$

Solution: $dv = \frac{\partial v}{\partial x}\cdot dx + \frac{\partial v}{\partial y}\cdot dy$

$$\Rightarrow \quad dv = -\frac{\partial u}{\partial y}\cdot dx + \underline{\frac{\partial u}{\partial x}\cdot dy}$$

$$dv = +e^x(x\cos y - y\sin y + \cos y)\cdot \underline{dy} -$$

$$e^x(-x\sin y - y\cos y - \sin y)\cdot dx$$

Integrating.

$$\int dv = \int e^x(x\sin y + y\cos y + \sin y)\cdot dx$$

(Treating y as constant

$$+\int e^x(x\cos y - y\sin y + \cos y)\cdot dy$$

(Those terms which do not contain x)

$$\Rightarrow v = \sin y \int x\cdot e^x\cdot dx + (y\cos y + \sin y)\int e^x\cdot dx + \int 0\cdot dy + C$$

$$\Rightarrow v = \sin y\cdot\{x\cdot e^x - e^x\} + (y\cos y + \sin y)\cdot e^x + C$$

$$\Rightarrow \boxed{v = (x\sin y + y\cos y)\cdot e^x + C}$$

Now, $f(z) = u + iv$

$$= e^x\cdot\{x\cos y - y\sin y + ix\sin y + iy\cos y\} + iC$$

$$= e^x\cdot\{x(\cos y + i\sin y) + iy(\cos y + i\sin y)\} + C'$$

$$= e^x\cdot(x+iy)(\cos y + i\sin y) + C'$$

$$\Rightarrow f(z) = (x+iy)\cdot e^x\cdot e^{iy} + c' \Rightarrow \boxed{f(z) = ze^z + c'}$$

complex form of C-R equations

Q.1 Let $p(z)$ and $q(z)$ be two non-zero complex polynomials then $p(z)\cdot\overline{q(z)}$ is analytic if and only if

(a) $p(z)$ is constant (b) $p(z)\cdot q(z)$ is constant

(c) $q(z)$ is constant (d) $\overline{p(z)}\,q(z)$ is constant

Solution: Let $f(z) = p(z)\cdot\overline{q(z)}$

$\because$ $f(z)$ is analytic

$\therefore$ $\boxed{\dfrac{df}{d\bar{z}} = 0} \Rightarrow \dfrac{d}{d\bar{z}}\{p(z)\cdot\overline{q(z)}\} = 0$

$\dfrac{d}{d\bar{z}}(2) = 0$

$\Rightarrow p(z)\cdot\dfrac{d}{d\bar{z}}\overline{q(z)} = 0$

$\Rightarrow \dfrac{d}{d\bar{z}}\overline{q(z)} = 0$ {given that $p(z) \neq 0$}

$\Rightarrow \overline{q(z)} = \text{Constant}$

$\dfrac{d}{d\bar{z}}f(z) = 0$

$\Rightarrow q(z) = \text{Constant}$

$f(z) = \text{const}$

Hence, option (c) is correct.

Let $\boxed{p(z) = z}$, $q(z) = 1 \Rightarrow \overline{q(z)} = 1$

Thus, $f(z) = p(z)\cdot\overline{q(z)} = z$ (Analytic).

But $p(z)$ is not constant.

Also $p(z)\cdot q(z) = z$ is also not constant.

and $\overline{p(z)}\cdot q(z) = \bar{z}$ is not constant.

Hence, (a), (b) and (d) are incorrect.

Q.2 Let $f: \mathbb{C} \to \mathbb{C}$ be a complex valued function given by $f(z) = u(x,y) + iv(x,y)$, suppose that $v(x,y) = 3xy^2$, then

(a) f can not be holomorphic on $\mathbb{C}$ for any choice of u.
(b) f is holomorphic on $\mathbb{C}$ for a suitable choice of u.
(c) f is holomorphic on $\mathbb{C}$ for all choices of u
(d) v is not diff^r as a function of x and y

Solution: Since, $v(x,y) = 3xy^2$

$$\Rightarrow v_x = 3y^2, \quad v_y = 6xy$$

$$\Rightarrow v_{xx} = 0, \quad v_{yy} = 6x.$$

Now, $v_{xx} + v_{yy} = 0 + 6x \neq 0$ (Not a pointwise property).

$\Rightarrow$ v is not harmonic

$\Rightarrow$ f is not analytic.

Hence, option (a) is correct.

NET/Gate

Q.3. Let $f: \mathbb{C} \to \mathbb{C}$ be a complex valued function of the form $f(x,y) = u(x,y) + iv(x,y)$. Suppose that $u(x,y) = 3x^2y$, then

(a) f can not be holomorphic for any choice of v.

(Same as above example).

70%

40/80

Q.4. Let $f: \mathbb{C} \to \mathbb{C}$ be an analytic function. For $z = x+iy$. Let $u, v: \mathbb{R}^2 \to \mathbb{R}$ be such that $u(x,y) = \text{Re}\, f(z)$ and $v(x,y) = \text{Im}\, f(z)$. Which of the following are correct?

(a) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ (b) $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

(c) $\dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial^2 u}{\partial y \partial x} = 0$ (d) $\dfrac{\partial^2 v}{\partial x \partial y} + \dfrac{\partial^2 v}{\partial y \partial x} = 0$

Solution: Since, f is analytic,

$\Rightarrow$ (i) C-R eq^n satisfies

(ii) u_x, u_y, v_x, v_y exist and continuous

(iii) Harmonic. (Both u and v)

$\star$ $$\frac{\partial^2 u}{\partial x \cdot \partial y} = \frac{\partial^2 u}{\partial y \cdot \partial x}$$ only when u is continuous.

Hence, (a), (b) and (c) are correct.

Q.5. Let $u(x,y) = x^3 - 3xy^2 + 2x$, For which of the following function v, $u + iv$ is holomorphic one?

(a) $v(x,y) = y^3 - 3x^2y + 2y$

(b) $v(x,y) = 3x^2y - y^3 + 2y$

(c) $v(x,y) = x^3 - 3xy^2 + 2x$

(d) $v(x,y) = 0$

C-R
- Harmonic

Solution: Given $u(x,y) = x^3 - 3xy^2 + 2x$.

$\Rightarrow u_x = 3x^2 - 3y^2 + 2, \quad u_y = -6xy$

$\because$ $u + iv$ is holomorphic.

$\therefore$ $u_x = v_y$ and $u_y = -v_x$ (C-R equation).

(i) $v_y = 3y^2 - 3x^2 + 2 \neq u_x$ option (a) is incorrect.

(ii) $v_y = 3x^2 - 3y^2 + 2 = u_x$
and $v_x = 6xy = -u_y$
$\Rightarrow$ C-R eq^n satisfied. option (b) is correct.

(iii) $v_y = -6xy \neq u_x$ option (c) is incorrect.

(iv) $v_y = 0 \neq u_x$ option (d) is incorrect.

Q.6 Let $f = u + iv$ be an entire function, where u and v are the real and imaginary parts of f respectively. If the Jacobian matrix

$$J_a = \begin{bmatrix} u_x(a) & u_y(a) \\ v_x(a) & v_y(a) \end{bmatrix}$$ is symmetric $\forall a \in \mathbb{C}$

then

(a) f is polynomial

(b) f is a polynomial of degree ≤ 1

(c) f is necessarily a constant function

(d) f is a polynomial of degree strictly greater than 1

$f(z) = az + b$

$f(z) = a_0 z + b$

$f(z) = a_0 z^2 + a_1 z + a_2$

C-R: $u_x = v_y$, $u_y = -v_x$

M.Sc: $u_y = v_x$, $-v_x = v_x$, $2v_x = 0$, $v_x = 0$, $v = f(y)$

Solution: Since, J_a is symmetric

$\Rightarrow u_y(a) = v_x(a)$ —①

Also f is analytic.

$\Rightarrow u_x = v_y$ and $u_y = -v_x$ —②

From ① and ②, we get $v_x = -v_x$

$\Rightarrow 2v_x = 0$

$\Rightarrow \boxed{v_x = 0}$

$\Rightarrow \boxed{u_y = 0}$ {from ②}

(i) $v_x = 0 \Rightarrow v = \text{Constant} + \psi(y)$

(ii) $u_y = 0 \Rightarrow u = \text{Constant} + \phi(x)$

(iii) For $f(z) = z$, $az+b$ $\quad J_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

All conditions are satisfied. (But f is not a constant)

(iv) If $\psi(y) = 0$ and $\phi(x) = 0$, then f is constant.

Hence, option (a) and (b) are correct.

any doubt ??

MSQ

options

Practice

Note

Manish

maneesh

K

Doubt

Practice